1. **Do not write your name on the exam. Your exam has a "secret student number" on each page. If you include any pages beyond those included with the exam, be sure to write that number on EACH additional page of the solutions you submit**.

2. This exam is closed-book, closed-mouth, closed-notes and closed-internet. You are not permitted to use mathematical software, e.g. mathematica. You are not to communicate with any other individuals regarding the exam, during the exam.

$$\int_{-\infty}^{\infty} dx\; e^{-x^2/(2a^2)} = a\sqrt{2\pi},$$

$$H = i\hbar\partial_t,\; \vec{P} = -i\hbar\nabla,$$

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

$$U(t, -\infty) = 1 + \frac{-i}{\hbar}\int_{-\infty}^{t} dt'\; V(t')U(t', -\infty),$$

$$\langle x|x'\rangle = \delta(x - x'),\; \langle p|p'\rangle = \frac{1}{2\pi\hbar}\delta(p - p'),$$

$$|p\rangle = \int dx\; |x\rangle e^{ipx/\hbar}, \quad |x\rangle = \int \frac{dp}{2\pi\hbar}|p\rangle e^{-ipx/\hbar},$$

$$H = \frac{P^2}{2m} + \frac{1}{2}m\omega^2 x^2 = \hbar\omega(a^\dagger a + 1/2),$$

$$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}X - i\sqrt{\frac{1}{2\hbar m\omega}}P,$$

$$\psi_0(x) = \frac{1}{(\pi b^2)^{1/4}}e^{-x^2/2b^2}, \quad b^2 = \frac{\hbar}{m\omega},$$

$$\rho(\vec{r}, t) = \psi^*(\vec{r}_1, t_1)\psi(\vec{r}_2, t_2)$$

$$\vec{j}(\vec{r}, t) = \frac{-i\hbar}{2m}(\psi^*(\vec{r}, t)\nabla\psi(\vec{r}, t) - (\nabla\psi^*(\vec{r}, t))\psi(\vec{r}, t)) - \frac{e\vec{A}}{mc}|\psi(\vec{r}, t)|^2.$$

$$H = \frac{(\vec{P} - e\vec{A}/c)^2}{2m} + e\Phi,$$

$$\text{For } V = \beta\delta(x - y): \quad -\frac{\hbar^2}{2m}\left(\frac{\partial}{\partial x}\psi(x)|_{y+\epsilon} - \frac{\partial}{\partial x}\psi(x)|_{y-\epsilon}\right) = -\beta\psi(y),$$

$$\vec{E} = -\nabla\Phi - \frac{1}{c}\partial_t\vec{A}, \quad \vec{B} = \nabla\times\vec{A},$$

$$\omega_{\text{cyclotron}} = \frac{eB}{mc},$$

$$e^{A+B} = e^A e^B e^{-C/2}, \quad \text{if } [A, B] = C, \text{ and } [C, A] = [C, B] = 0,$$

$$Y_{0,0} = \frac{1}{\sqrt{4\pi}}, \quad Y_{1,0} = \sqrt{\frac{3}{4\pi}}\cos\theta, \quad Y_{1,\pm1} = \mp\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\pm\phi},$$

$$Y_{2,0} = \sqrt{\frac{5}{16\pi}}(3\cos^2\theta - 1), \quad Y_{2,\pm1} = \mp\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta e^{\pm i\phi},$$

$$Y_{2,\pm2} = \sqrt{\frac{15}{32\pi}}\sin^2\theta e^{\pm 2i\phi}, \quad Y_{\ell -m}(\theta, \phi) = (-1)^m Y^*_{\ell m}(\theta, \phi).$$

$$|N\rangle = |n\rangle - \sum_{m\neq n} |m\rangle \frac{1}{\epsilon_m - \epsilon_n} \langle m|V|n\rangle + \cdots$$

$$E_N = \epsilon_n + \langle n|V|n\rangle - \sum_{m\neq n} \frac{|\langle m|V|n\rangle|^2}{\epsilon_m - \epsilon_n}$$

$$j_0(x) = \frac{\sin x}{x},\ n_0(x) = -\frac{\cos x}{x},\ j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x},\ n_1(x) = -\frac{\cos x}{x^2} - \frac{\sin x}{x}$$

$$j_2(x) = \left(\frac{3}{x^3} - \frac{1}{x}\right)\sin x - \frac{3}{x^2}\cos x,\ n_2(x) = -\left(\frac{3}{x^3} - \frac{1}{x}\right)\cos x - \frac{3}{x^2}\sin x,$$

$$\frac{d}{dt}P_{i\to n}(t) = \frac{2\pi}{\hbar}|V_{ni}|^2\delta(E_n - E_i),$$

$$\frac{d\sigma}{d\Omega} = \frac{m^2}{4\pi^2\hbar^4}\left|\int d^3r\mathcal{V}(r)e^{i(\vec{k}_f - \vec{k}_i)\cdot\vec{r}}\right|^2,$$

$$\sigma = \frac{(2S_R+1)}{(2S_1+1)(2S_2+1)}\frac{4\pi}{k^2}\frac{(\hbar\Gamma_R/2)^2}{(\epsilon_k - \epsilon_r)^2 + (\hbar\Gamma_R/2)^2},$$

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_{\text{single}}\tilde{S}(\vec{q}),\quad \tilde{S}(\vec{q}) = \left|\sum_{\vec{a}} e^{i\vec{q}\cdot\vec{a}}\right|^2,$$

$$e^{i\vec{k}\cdot\vec{r}} = \sum_\ell (2\ell+1)i^\ell j_\ell(kr)P_\ell(\cos\theta),$$

$$P_\ell(\cos\theta) = \sqrt{\frac{4\pi}{2\ell+1}}Y_{\ell,m=0}(\theta,\phi),$$

$$P_0(x) = 1,\ P_1(x) = x,\ P_2(x) = (3x^2-1)/3,$$

$$f(\Omega) \equiv \sum_\ell (2\ell+1)e^{i\delta_\ell}\sin\delta_\ell \frac{1}{k}P_\ell(\cos\theta)$$

$$\psi_{\vec{k}}(\vec{r})|_{R\to\infty} = e^{i\vec{k}\cdot\vec{r}} + \frac{e^{ikr}}{r}f(\Omega),$$

$$\frac{d\sigma}{d\Omega} = |f(\Omega)|^2,\quad \sigma = \frac{4\pi}{k^2}\sum_\ell (2\ell+1)\sin^2\delta_\ell,\quad \delta \approx -ak$$

$$L_\pm|\ell,m\rangle = \sqrt{\ell(\ell+1) - m(m\pm 1)}|\ell,m\pm 1\rangle,$$

$$C^{\ell,s}_{m_\ell,m_s;JM} = \langle \ell,s,J,M|\ell,s,m_\ell,m_s\rangle,$$

$$\langle\tilde{\beta},J,M|T^k_q|\beta,\ell,m_\ell\rangle = C^{k\ell}_{qm_\ell;JM}\frac{\langle\tilde{\beta},J||T^{(k)}||\beta,\ell,J\rangle}{\sqrt{2J+1}},$$

$$n = \frac{(2s+1)}{(2\pi)^d}\int_{k<k_f} d^dk,\quad d\text{ dimensions},$$

$$\{\Psi_s(\vec{x}),\Psi^\dagger_{s'}(\vec{y})\} = \delta^3(\vec{x}-\vec{y})\delta_{ss'},$$

$$\Psi^\dagger_s(\vec{r}) = \frac{1}{\sqrt{V}}\sum_{\vec{k}} e^{i\vec{k}\cdot\vec{r}}a^\dagger_s(\vec{k}),\quad \{\Psi_s(\vec{x}),a^\dagger_\alpha\} = \phi_{\alpha,s}(\vec{x}).$$

1. Consider a two-component system with the original Hamiltonian,

$$\boldsymbol{H_0 = A\sigma_z, \qquad A > 0.}$$

A time $\boldsymbol{t = 0}$ one adds an additional potential

$$\boldsymbol{V = g\sigma_x.}$$

- (10 pts) If at time $\boldsymbol{t = 0}$ the system is in the ground state of $\boldsymbol{H_0}$, find the probability for being in that original ground state as a function of time.
- (10 pts) To second order in perturbation theory (where $\boldsymbol{V}$ is the perturbation), what is the ground state energy?
- (10 pts) What is the exact solution for the eigen-energies of the full Hamiltonian?

a) $\varepsilon_0 = -A, \quad \varepsilon_1 = A$

$$U(t) = \exp\{-i(A\sigma_z + V\sigma_x)t/\hbar\}$$

$$= \exp\{-i\beta\, \vec{\sigma}\cdot\hat{n}\, t\}$$

$$\beta = \sqrt{A^2 + V^2}/\hbar$$

$$\hat{n} = \frac{1}{\sqrt{A^2+V^2}}\left(A\hat{z} + V\hat{x}\right)$$

$$U = \cos\beta t - i\,\vec{\sigma}\cdot\hat{n}\,\sin\beta t$$

$$P_0(t) = \left|\begin{pmatrix}0\\1\end{pmatrix}^{+} U(t)\begin{pmatrix}0\\1\end{pmatrix}\right|^2 = \left|\cos\beta t + \frac{iA}{\sqrt{A^2+V^2}}\sin\beta t\right|^2$$

$$= \cos^2\beta t + \frac{A^2}{A^2+V^2}\sin^2\beta t$$

b) $$\delta E^{(2)} = -\frac{|\langle 1|V|0\rangle|^2}{\varepsilon_1 - \varepsilon_0} = -\frac{g^2}{2A}$$

c) $$E = \pm\sqrt{A^2 + V^2}$$

2. Consider the states

$$|\boldsymbol{\eta}\rangle = \mathbf{exp}\left\{\boldsymbol{i}(\boldsymbol{\eta a}^\dagger + \boldsymbol{\eta}^* \boldsymbol{a})\right\}|\mathbf{0}\rangle,$$
$$|\gamma\rangle = \mathbf{exp}\left\{\boldsymbol{i}(\gamma \boldsymbol{a}^\dagger + \gamma^* \boldsymbol{a})\right\}|\mathbf{0}\rangle,$$

Here, $\boldsymbol{\eta}$ and $\boldsymbol{\gamma}$ are complex numbers, and $\boldsymbol{a}^\dagger$ and $\boldsymbol{a}$ are creation and annihilation operators respectively.

- (10 pts) Calculate $\langle \boldsymbol{\eta}|\boldsymbol{\eta}\rangle$.
- (10 pts) Calculate $\langle \gamma|\boldsymbol{\eta}\rangle$.

a) $\langle \eta | \eta \rangle = 1$, because $(\eta a^\dagger + \eta^* a)$ is Hermitian

Baker Campbell Hausdorff

b) $|\eta\rangle = e^{-|\eta|^2/2} e^{i\eta a^\dagger} |0\rangle$

$|\gamma\rangle = e^{-|\gamma|^2/2} e^{i\gamma a^\dagger} |0\rangle$

$\langle \gamma | \eta \rangle = e^{-|\eta|^2/2 - |\gamma|^2/2} \langle 0 | e^{-i\gamma^* a} e^{i\eta a^\dagger} |0\rangle$

$a|\eta\rangle = i\eta a$

$\langle \gamma | \eta \rangle = e^{-|\eta|^2/2 - |\gamma|^2/2} e^{\gamma^* \eta} \langle 0 | e^{i\eta a^\dagger} |0\rangle$

$= e^{-|\eta|^2/2 - |\gamma|^2/2} e^{\gamma^* \eta}$

3. (20 pts) Two point charges are positioned in a line along the $\boldsymbol{z}$ axis, with a distance $\boldsymbol{a}$ separating the charges. A beam with momentum $\boldsymbol{p}$ is incident on the charges along the $\boldsymbol{z}$ axis. The scattering can be considered as a perturbative process. In order to determine the distance $\boldsymbol{a}$, you measure the directions at which the differential cross section is the smallest. In terms of $\boldsymbol{a}$ and $\boldsymbol{p}$, list the angles for which all the scattering is smallest. Use $\boldsymbol{\theta}$ for the polar angle, the angle relative to the $\boldsymbol{z}$ axis and $\boldsymbol{\phi}$ for the azimuthal angle.

$$\vec{q} = p\hat{z} - p\hat{z}\cos\theta - p\hat{x}\sin\theta\cos\phi - p\hat{y}\sin\theta\sin\phi$$

$$\frac{d\sigma}{d\Omega} \sim \left|1 + e^{i\vec{q}\cdot\vec{a}}\right|^2$$

$$= \left|1 + e^{ipa(1-\cos\theta)/\hbar}\right|$$

$$= 0 \quad \text{when } \frac{1}{\hbar}pa(1-\cos\theta) = \pi, 3\pi, 5\pi \cdots$$

$$1 - \cos\theta_n = \frac{(2n+1)\pi\hbar}{pa}$$

$$\theta_n = \arccos\left\{1 - \frac{(2n+1)\pi\hbar}{pa}\right\}$$

works until $\theta_n > \pi$

Extra space for #3

4. Consider a particle of mass m in a three-dimensional potential

$$V(r) = -\beta\delta(r-a), \qquad \beta > 0.$$

- (10 pts) In terms of β and m, what is the minimum value of a for which one has a bound state?
- (10 pts) Assuming β is above the value above, what is the $s-$wave phase shift as a function of the magnitude of the momentum, $\hbar k$?

a) $$-\frac{\hbar^2}{2m}\partial_r^2\psi = \left[E + \beta\delta(r-a)\right]\psi.$$

$$-\partial_r\psi(a+\varepsilon) + \partial_r\psi(a-\varepsilon) = \frac{2m\beta}{\hbar^2}\psi(a)$$

$$\psi_I = \sinh\kappa r, \quad \kappa \to 0$$

$$\psi_{II} = A\exp(-\kappa r)$$

$$\sinh\kappa a = A e^{-\kappa a}$$

$$A\kappa e^{-\kappa a} - \kappa\cosh\kappa a = \frac{2m\beta}{\hbar^2}\sinh\kappa a, \quad \kappa \to 0$$

$$\kappa\sin\cancel{h}\kappa a - \kappa\cosh\kappa a = \frac{2m\beta}{\hbar^2}\sinh\kappa a$$

$$1 = \frac{2m\beta a}{\hbar^2}$$

$$a = \frac{\hbar^2}{2m\beta}$$

b) $$\psi_I = A\sin kr, \quad \psi_{II} = \sin(kr+\delta)$$

$$A\sin ka = \sin(ka+\delta)$$

$$kA\cos ka = k\cos(ka+\delta) + \frac{2m\beta}{\hbar^2}A\sin ka$$

$$\frac{\sin ka}{k\left(\cos ka - \frac{2m\beta}{\hbar^2}\sin ka\right)} = \frac{1}{k}\tan(ka+\delta)$$

$$\delta = -ka + \arctan\left\{\frac{\sin ka}{\cos ka - \frac{2m\beta}{\hbar^2 k}\sin ka}\right\}$$

Extra space for #4

5. (30 pts) Consider a ONE-DIMENSIONAL world, where a non-relativistic particle of mass M is in the ground state of a harmonic oscillator characterized by frequency ω_0. These massive particles are created and destroyed with field operators, $\Psi(x)$, which obey the commutation relations

$$[\Psi(x), \Psi^\dagger(y)] = \delta(x - y).$$

These particles, referred to as Ψ particles, can transform into Φ particles, which have exactly the same mass. However, the Φ particles do not feel the harmonic oscillator potential. The perturbative interaction responsible for the transformation is

$$V = g \int dx \left(\Psi^\dagger(x)\Phi(x) + \Phi^\dagger(x)\Psi(x)\right)$$

Using Fermi's golden rule, calculate the rate of decay of Ψ particles into Φ particles.

$$\psi_0(x) = \frac{1}{(\pi b^2)^{1/4}} e^{-x^2/2b^2}, \quad b^2 \equiv \hbar/m\omega$$

$$\Gamma = \frac{2\pi}{\hbar} \sum_k |\langle k|V|0\rangle|^2 \delta(\varepsilon_0 - \varepsilon_k)$$

$$\langle k|V|0\rangle = \int \frac{e^{-ikx}}{\sqrt{L}} \cdot g \cdot \psi_0(x)\, dx$$

complete square

$$= \frac{g}{\sqrt{L}} \cdot \frac{1}{(\pi b^2)^{1/4}} \cdot \int dx\, e^{-x^2/2b^2 - ikx}$$

$$= \frac{g}{\sqrt{L}} \frac{1}{(\pi b^2)^{1/4}} e^{-k^2b^2/2} (2\pi b^2)^{1/2}$$

$$= \frac{g}{2^{1/2}} \pi^{1/4} b^{1/2} e^{-k^2b^2/2}$$

$$\Gamma = \frac{2\pi}{\hbar} \frac{g^2}{2} \pi^{1/2} b\, e^{-k^2b^2} \sum_k \delta(\varepsilon_0 - \varepsilon_k)$$

$$= \frac{2\pi}{\hbar} \frac{g^2}{\cancel{2}} \pi^{1/2} b\, e^{-k^2b^2} \frac{L}{\pi} \int_0^\infty dk\, \delta(\varepsilon_0 - \varepsilon_k)$$

both $\pm k$

$$= \frac{2\pi^{1/2}}{\hbar} g^2 b\, e^{-k^2b^2} \frac{1}{\frac{d}{dk}\frac{\hbar^2k^2}{2m}} = \frac{2m\pi^{1/2}}{\hbar^3 k} g^2 b\, e^{-k^2b^2}$$

Extra space for #5

$$\Gamma = \frac{2m\pi^{1/2} g^2}{\hbar^3 k}\left(\frac{\hbar}{m\omega}\right)^{1/2} e^{-k^2 b^2}$$

$$\hbar^2 k^2 = 2m\hbar\omega/2$$

$$\hbar k = (\hbar\omega m)^{1/2}$$

$$\Gamma = \frac{2m\pi^{1/2} g^2}{\hbar^2 (\hbar m\omega)^{1/2}}\left(\frac{\hbar}{m\omega}\right)^{1/2} e^{-k^2 b^2}$$

$$= \frac{2\pi^{1/2} g^2}{\hbar^2 \omega} e^{-k^2 b^2}, \quad k^2 b^2 = \frac{\hbar}{m\omega}\frac{2m(\hbar\omega/2)}{\hbar^2}$$

$$= 1$$

$$= \frac{2\pi^{1/2} g^2}{\hbar^2 \omega} e^{-1}$$