MIDTERM PRACTICE EXAM,

PHYSICS 852, Spring 2020

Friday, Feb. 28, 1:50-2:40 PM

This practice exam is worth 0 points

SECRET STUDENT NUMBER:
STUDNUMBER

$$\int_{-\infty}^{\infty} dx\ e^{-x^2/2} = \sqrt{2\pi},$$

$$H = i\hbar\partial_t,\ \vec{P} = -i\hbar\nabla,$$

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

$$U(t,-\infty) = 1 + \frac{-i}{\hbar}\int_{-\infty}^{t} dt'\ V(t')U(t',-\infty),$$

$$\langle x|x'\rangle = \delta(x-x'),\ \langle p|p'\rangle = \frac{1}{2\pi\hbar}\delta(p-p'),$$

$$|p\rangle = \int dx\ |x\rangle e^{ipx/\hbar},\quad |x\rangle = \int \frac{dp}{2\pi\hbar}|p\rangle e^{-ipx/\hbar},$$

$$H = \frac{P^2}{2m} + \frac{1}{2}m\omega^2x^2 = \hbar\omega(a^\dagger a + 1/2),$$

$$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}X - i\sqrt{\frac{1}{2\hbar m\omega}}P,$$

$$\psi_0(x) = \frac{1}{(\pi b^2)^{1/4}}e^{-x^2/2b^2},\quad b^2 = \frac{\hbar}{m\omega},$$

$$\rho(\vec{r},t) = \psi^*(\vec{r}_1,t_1)\psi(\vec{r}_2,t_2)$$

$$\vec{j}(\vec{r},t) = \frac{-i\hbar}{2m}(\psi^*(\vec{r},t)\nabla\psi(\vec{r},t) - (\nabla\psi^*(\vec{r},t))\psi(\vec{r},t)) - \frac{e\vec{A}}{mc}|\psi(\vec{r},t)|^2.$$

$$H = \frac{(\vec{P}-e\vec{A}/c)^2}{2m} + e\Phi,$$

$$\text{For } V = \beta\delta(x-y):\quad -\frac{\hbar^2}{2m}\left(\frac{\partial}{\partial x}\psi(x)|_{y+\epsilon} - \frac{\partial}{\partial x}\psi(x)|_{y-\epsilon}\right) = -\beta\psi(y),$$

$$\vec{E} = -\nabla\Phi - \frac{1}{c}\partial t\vec{A},\quad \vec{B} = \nabla\times\vec{A},$$

$$\omega_{\rm cyclotron} = \frac{eB}{mc},$$

$$e^{A+B} = e^Ae^Be^{-C/2},\ \text{ if } [A,B]=C,\ \text{ and } [C,A]=[C,B]=0,$$

$$Y_{0,0} = \frac{1}{\sqrt{4\pi}},\quad Y_{1,0} = \sqrt{\frac{3}{4\pi}}\cos\theta,\quad Y_{1,\pm1} = \mp\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\pm\phi},$$

$$Y_{2,0} = \sqrt{\frac{5}{16\pi}}(3\cos^2\theta - 1),\quad Y_{2,\pm1} = \mp\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta e^{\pm i\phi},$$

$$Y_{2,\pm2} = \sqrt{\frac{15}{32\pi}}\sin^2\theta e^{\pm 2i\phi},\quad Y_{\ell -m}(\theta,\phi) = (-1)^m Y^*_{\ell m}(\theta,\phi).$$

$$|N\rangle = |n\rangle - \sum_{m\neq n} |m\rangle \frac{1}{\epsilon_m - \epsilon_n} \langle m|V|n\rangle + \cdots$$

$$E_N = \epsilon_n + \langle n|V|n\rangle - \sum_{m\neq n} \frac{|\langle m|V|n\rangle|^2}{\epsilon_m - \epsilon_n}$$

$$j_0(x) = \frac{\sin x}{x},\ n_0(x) = -\frac{\cos x}{x},\ j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x},\ n_1(x) = -\frac{\cos x}{x^2} - \frac{\sin x}{x}$$

$$j_2(x) = \left(\frac{3}{x^3} - \frac{1}{x}\right)\sin x - \frac{3}{x^2}\cos x,\ n_2(x) = -\left(\frac{3}{x^3} - \frac{1}{x}\right)\cos x - \frac{3}{x^2}\sin x,$$

$$\frac{d}{dt}P_{i\to n}(t) = \frac{2\pi}{\hbar}|V_{ni}|^2\delta(E_n - E_i),$$

$$\frac{d\sigma}{d\Omega} = \frac{m^2}{4\pi^2\hbar^4}\left|\int d^3r \mathcal{V}(r) e^{i(\vec{k}_f - \vec{k}_i)\cdot\vec{r}}\right|^2,$$

$$\sigma = \frac{(2S_R+1)}{(2S_1+1)(2S_2+1)}\frac{4\pi}{k^2}\frac{(\hbar\Gamma_R/2)^2}{(\epsilon_k - \epsilon_r)^2 + (\hbar\Gamma_R/2)^2},$$

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_{\text{single}}\tilde{S}(\vec{q}),\quad \tilde{S}(\vec{q}) = \left|\sum_{\delta\vec{a}} e^{i\vec{q}\cdot\delta\vec{a}}\right|^2,$$

$$e^{i\vec{k}\cdot\vec{r}} = \sum_\ell (2\ell+1) i^\ell j_\ell(kr) P_\ell(\cos\theta),$$

$$P_\ell(\cos\theta) = \sqrt{\frac{4\pi}{2\ell+1}} Y_{\ell,m=0}(\theta,\phi),$$

$$P_0(x) = 1,\ P_1(x) = x,\ P_2(x) = (3x^2-1)/3,$$

$$f(\Omega) \equiv \sum_\ell (2\ell+1) e^{i\delta_\ell}\sin\delta_\ell \frac{1}{k} P_\ell(\cos\theta)$$

$$\psi_{\vec{k}}(\vec{r})|_{R\to\infty} = e^{i\vec{k}\cdot\vec{r}} + \frac{e^{ikr}}{r} f(\Omega),$$

$$\frac{d\sigma}{d\Omega} = |f(\Omega)|^2,\quad \sigma = \frac{4\pi}{k^2}\sum_\ell (2\ell+1)\sin^2\delta_\ell,$$

$$L_\pm|\ell,m\rangle = \sqrt{\ell(\ell+1) - m(m\pm 1)}|\ell, m\pm 1\rangle,$$

$$C^{\ell,s}_{m_\ell,m_s;JM} = \langle \ell,s,J,M|\ell,s,m_\ell,m_s\rangle,$$

$$\langle\tilde{\beta},J,M|T^k_q|\beta,\ell,m_\ell\rangle = C^{k\ell}_{qm_\ell;JM}\frac{\langle\tilde{\beta},J||T^{(k)}||\beta,\ell,J\rangle}{\sqrt{2J+1}},$$

$$n = \frac{(2s+1)}{(2\pi)^d}\int_{k<k_f} d^dk,\quad d \text{ dimensions},$$

$$\{\Psi_s(\vec{x}),\Psi^\dagger_{s'}(\vec{y})\} = \delta^3(\vec{x}-\vec{y})\delta_{ss'},$$

$$\Psi^\dagger_s(\vec{r}) = \frac{1}{\sqrt{V}}\sum_{\vec{k}} e^{i\vec{k}\cdot\vec{r}} a^\dagger_s(\vec{k}),\quad \{\Psi_s(\vec{x}), a^\dagger_\alpha\} = \phi_{\alpha,s}(\vec{x}).$$

1. Consider a one-dimensional world where a type-$\boldsymbol{A}$ particle of mass $\boldsymbol{M}$ has zero momentum and decays to two type-$\boldsymbol{B}$ particles. The type-$\boldsymbol{B}$ particles are massless. The interaction is of the form

$$\boldsymbol{V = g \int dx \; (\Psi_B^\dagger(x) + \Psi_B(x))^2 [\Phi_A(x) + \Phi_A^\dagger(x)]}.$$

The field operators are defined by

$$\boldsymbol{\Phi_A^\dagger(x) = \frac{1}{\sqrt{L}} \sum_k e^{-ikx} a_k^\dagger, \quad \Psi_B^\dagger(x) = \frac{1}{\sqrt{L}} \sum_k \frac{1}{\sqrt{E_k}} e^{-ikx} b_k^\dagger}.$$

Calculate the rate at which the type-$\boldsymbol{A}$ particle decays into a type-$\boldsymbol{B}$ particle.

$$\langle f | H_{int} | i \rangle = g \int dx \langle k, k' | \Psi_B^\dagger(x)^2 | 0 \rangle \cdot \langle 0 | \Phi_A(x) | k_A = 0 \rangle$$

$$= \frac{g \int dx \; e^{-i(k+k')\cdot x}}{L^{3/2} E_k} \, (2) = \frac{2g}{L^{1/2} E_k} \delta_{k,-k'}$$

← 2 ways to connect $\Psi_B^{\dagger 2}$ to k, k'

$$\delta(E_k + E_{k'} - Mc^2)$$

$$\Gamma_A = \frac{2\pi}{\hbar} \frac{4g^2}{L E_k^2} \sum_{k,k'} \delta_{k-k'}$$

$$= \frac{2\pi}{\hbar} \frac{4g^2}{E_k^2 L} \frac{L}{2\pi} \int_0^\infty dk \; \delta(2E_k - Mc^2)$$

$$\delta(2\hbar c k - Mc^2)$$

$$= \frac{2\pi}{\hbar} \frac{4g^2}{(\hbar c k)^2} \frac{1}{2\pi} \frac{1}{2\hbar c}$$

$$\hbar c k = \frac{Mc^2}{2}$$

$$= \frac{4g^2}{2\hbar^4 c^3 k^2}$$

$$= \frac{8g^2}{\hbar^2 M^2 c^5}$$

(Extra work space for #1)

2. Consider the following matrix element,

$$\mathcal{M}(m_i, m_f) = \langle \ell_f = 2, m_f | P_x^2 + P_y^2 | \ell_i = 4, m_i \rangle.$$

(a) For which combinations of m_i, m_f is $\mathcal{M}$ non-zero?

(b) If one were to calculate the matrix element

$$\mathcal{M}(m_i = 0, m_f = 0) = \langle \ell_f = 2, m_f = 0 | P_x^2 + P_y^2 | \ell_i = 4, m_i = 0 \rangle,$$

express the non-zero elements from (a) in terms of $\mathcal{M}(0,0)$. You can leave the answer in terms of ratios of Clebsch-Gordan coefficients.

a)

m_i	m_f
2	2
1	1
0	0
-1	-1
-2	-2

b) $$M(m_i\ m_f) = \frac{\langle 2\, m_f | 2\, 0\, 4\, m_i \rangle}{\langle 2\, 0 | 2\, 0\, 4\, 0 \rangle} \cdot M(0,0)$$

(Extra work space for #2)

3. Electrons are confined to a two-dimensional surface to move in the $x - y$ plane ($z = p_z = 0$). The areal density of electrons, number of electrons per area, is σ. Electrons of the same spin have the same energy until a magnetic field is added along the z axis. This gives the interaction,

$$H_B = g_s \mu_B B \frac{s_z}{\hbar},$$

where $\mu_B = e\hbar/2mc$, is the Bohr magneton and $g_s = 2$. Note the sign is positive above because the magnetic moment of the electron is negative (due to its negative charge).

(a) In terms of m and σ, what are the Fermi energy, ϵ_{f0}, and the Fermi wave number, k_{f0}, when $B = 0$?

(b) What is the aerial magnetic moment density, M_z, for small fields?

$$M_z = \frac{1}{2} g_s \mu_B (\sigma_\downarrow - \sigma_\uparrow) = \chi B.$$

where $\sigma_\uparrow$ is the aerial density of spin-up electrons. Express χ in terms of $g_s, e, \hbar, m, c$ and k_f.

a) $\sigma = \frac{2}{4\pi} k_f^2$, $k_f = \sqrt{2\pi\sigma}$

$\epsilon_f = \frac{\hbar^2 \pi \sigma}{m}$

b) $\Delta\sigma_\uparrow = \frac{2}{4\pi} k_f \frac{\Delta k_f}{2} = \frac{1}{4\pi} k_f \Delta k_f$

$\sigma_\uparrow - \sigma_\downarrow = \frac{1}{2\pi} k_f \Delta k_f$

$\Delta \epsilon_f = 2\mu_B B = \frac{\hbar^2 k_f}{m} \Delta k_f$

$\sigma_\uparrow - \sigma_\downarrow = \frac{1}{2\pi} k_f \frac{m}{\hbar^2 k_f} \cdot 2\mu_B B$

$= \frac{m}{\pi\hbar^2} \mu_B B$

$\chi = \frac{m \mu_B^2}{\pi \hbar^2} = \frac{e^2}{4\pi m c^2}$

(Extra work space for #3)