MIDTERM EXAM,

PHYSICS 852, Spring 2021

Friday, March 5, 1:50-2:40 PM

SECRET STUDENT NUMBER:
STUDNUMBER

$$\int_{-\infty}^{\infty} dx\ e^{-x^2/(2a^2)} = a\sqrt{2\pi},$$

$$H = i\hbar\partial_t,\ \vec{P} = -i\hbar\nabla,$$

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

$$U(t,-\infty) = 1 + \frac{-i}{\hbar}\int_{-\infty}^{t} dt'\ V(t')U(t',-\infty),$$

$$\langle x|x'\rangle = \delta(x-x'),\ \langle p|p'\rangle = \frac{1}{2\pi\hbar}\delta(p-p'),$$

$$|p\rangle = \int dx\ |x\rangle e^{ipx/\hbar},\quad |x\rangle = \int \frac{dp}{2\pi\hbar}|p\rangle e^{-ipx/\hbar},$$

$$H = \frac{P^2}{2m} + \frac{1}{2}m\omega^2x^2 = \hbar\omega(a^\dagger a + 1/2),$$

$$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}X - i\sqrt{\frac{1}{2\hbar m\omega}}P,$$

$$\psi_0(x) = \frac{1}{(\pi b^2)^{1/4}}e^{-x^2/2b^2},\quad b^2 = \frac{\hbar}{m\omega},$$

$$\rho(\vec{r},t) = \psi^*(\vec{r}_1,t_1)\psi(\vec{r}_2,t_2)$$

$$\vec{j}(\vec{r},t) = \frac{-i\hbar}{2m}(\psi^*(\vec{r},t)\nabla\psi(\vec{r},t) - (\nabla\psi^*(\vec{r},t))\psi(\vec{r},t)) - \frac{e\vec{A}}{mc}|\psi(\vec{r},t)|^2.$$

$$H = \frac{(\vec{P} - e\vec{A}/c)^2}{2m} + e\Phi,$$

For $V = \beta\delta(x-y):$

$$-\frac{\hbar^2}{2m}\left(\frac{\partial}{\partial x}\psi(x)|_{y+\epsilon} - \frac{\partial}{\partial x}\psi(x)|_{y-\epsilon}\right) = -\beta\psi(y),$$

$$\vec{E} = -\nabla\Phi - \frac{1}{c}\partial_t\vec{A},\quad \vec{B} = \nabla\times\vec{A},$$

$$\omega_{\rm cyclotron} = \frac{eB}{mc},$$

$$e^{A+B} = e^Ae^Be^{-C/2},\quad \text{if } [A,B] = C,\ \text{and } [C,A] = [C,B] = 0,$$

$$Y_{0,0} = \frac{1}{\sqrt{4\pi}},\quad Y_{1,0} = \sqrt{\frac{3}{4\pi}}\cos\theta,\quad Y_{1,\pm1} = \mp\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\pm\phi},$$

$$Y_{2,0} = \sqrt{\frac{5}{16\pi}}(3\cos^2\theta - 1),\quad Y_{2,\pm1} = \mp\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta e^{\pm i\phi},$$

$$Y_{2,\pm2} = \sqrt{\frac{15}{32\pi}}\sin^2\theta e^{\pm2i\phi},\quad Y_{\ell-m}(\theta,\phi) = (-1)^mY^*_{\ell m}(\theta,\phi).$$

$$|N\rangle = |n\rangle - \sum_{m\neq n} |m\rangle \frac{1}{\epsilon_m - \epsilon_n} \langle m|V|n\rangle + \cdots$$

$$E_N = \epsilon_n + \langle n|V|n\rangle - \sum_{m\neq n} \frac{|\langle m|V|n\rangle|^2}{\epsilon_m - \epsilon_n}$$

$$j_0(x) = \frac{\sin x}{x},\ n_0(x) = -\frac{\cos x}{x},\ j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x},\ n_1(x) = -\frac{\cos x}{x^2} - \frac{\sin x}{x}$$

$$j_2(x) = \left(\frac{3}{x^3} - \frac{1}{x}\right)\sin x - \frac{3}{x^2}\cos x,\ n_2(x) = -\left(\frac{3}{x^3} - \frac{1}{x}\right)\cos x - \frac{3}{x^2}\sin x,$$

$$\frac{d}{dt}P_{i\to n}(t) = \frac{2\pi}{\hbar}|V_{ni}|^2\delta(E_n - E_i),$$

$$\frac{d\sigma}{d\Omega} = \frac{m^2}{4\pi^2\hbar^4}\left|\int d^3r\mathcal{V}(r)e^{i(\vec{k}_f - \vec{k}_i)\cdot\vec{r}}\right|^2,$$

$$\sigma = \frac{(2S_R+1)}{(2S_1+1)(2S_2+1)}\frac{4\pi}{k^2}\frac{(\hbar\Gamma_R/2)^2}{(\epsilon_k - \epsilon_r)^2 + (\hbar\Gamma_R/2)^2},$$

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_{\rm single}\tilde{S}(\vec{q}),\quad \tilde{S}(\vec{q}) = \left|\sum_{\vec{a}} e^{i\vec{q}\cdot\vec{a}}\right|^2,$$

$$e^{i\vec{k}\cdot\vec{r}} = \sum_\ell (2\ell+1)i^\ell j_\ell(kr)P_\ell(\cos\theta),$$

$$P_\ell(\cos\theta) = \sqrt{\frac{4\pi}{2\ell+1}}Y_{\ell,m=0}(\theta,\phi),$$

$$P_0(x) = 1,\ P_1(x) = x,\ P_2(x) = (3x^2-1)/3,$$

$$f(\Omega) \equiv \sum_\ell (2\ell+1)e^{i\delta_\ell}\sin\delta_\ell\frac{1}{k}P_\ell(\cos\theta)$$

$$\psi_{\vec{k}}(\vec{r})|_{R\to\infty} = e^{i\vec{k}\cdot\vec{r}} + \frac{e^{ikr}}{r}f(\Omega),$$

$$\frac{d\sigma}{d\Omega} = |f(\Omega)|^2,\quad \sigma = \frac{4\pi}{k^2}\sum_\ell (2\ell+1)\sin^2\delta_\ell,\quad \delta \approx -ak$$

$$L_\pm|\ell,m\rangle = \sqrt{\ell(\ell+1) - m(m\pm1)}|\ell,m\pm1\rangle,$$

$$C^{\ell,s}_{m_\ell,m_s;JM} = \langle \ell,s,J,M|\ell,s,m_\ell,m_s\rangle,$$

$$\langle\tilde{\beta},J,M|T^k_q|\beta,\ell,m_\ell\rangle = C^{k\ell}_{qm_\ell;JM}\frac{\langle\tilde{\beta},J||T^{(k)}||\beta,\ell,J\rangle}{\sqrt{2J+1}},$$

$$n = \frac{(2s+1)}{(2\pi)^d}\int_{k<k_f} d^dk,\quad d \text{ dimensions},$$

$$\{\Psi_s(\vec{x}),\Psi^\dagger_{s'}(\vec{y})\} = \delta^3(\vec{x}-\vec{y})\delta_{ss'},$$

$$\Psi^\dagger_s(\vec{r}) = \frac{1}{\sqrt{V}}\sum_{\vec{k}} e^{i\vec{k}\cdot\vec{r}}a^\dagger_s(\vec{k}),\quad \{\Psi_s(\vec{x}),a^\dagger_\alpha\} = \phi_{\alpha,s}(\vec{x}).$$

1. Type α, β and γ particles exist in a TWO-DIMENSIONAL world. The α particle has mass M_α and is described by the two-dimensional field operator within a large area A,

$$\Phi_\alpha(\vec{r},t) = \frac{1}{\sqrt{A}}\sum_{\vec{k}} e^{-iE_k t/\hbar + i\vec{k}\cdot\vec{r}} a_{\vec{k}},$$

$$\Phi^\dagger_\alpha(\vec{r},t) = \frac{1}{\sqrt{A}}\sum_{\vec{k}} e^{iE_k t/\hbar - i\vec{k}\cdot\vec{r}} a^\dagger_{\vec{k}},$$

$$E_k = M_\alpha c^2 + \frac{\hbar^2 k^2}{2M_\alpha}.$$

The β and γ particles are massless and described by the operators,

$$\Psi_\beta(\vec{r},t) = \frac{1}{\sqrt{A}}\sum_{\vec{q}} e^{-iE_q t/\hbar + i\vec{q}\cdot\vec{r}} b_q,$$

$$\Psi^\dagger_\beta(\vec{r},t) = \frac{1}{\sqrt{A}}\sum_{\vec{q}} e^{iE_q t/\hbar - i\vec{q}\cdot\vec{r}} b^\dagger_q,$$

$$\Psi_\gamma(\vec{r},t) = \frac{1}{\sqrt{A}}\sum_{\vec{q}} e^{-iE_q t/\hbar + i\vec{q}\cdot\vec{r}} c_q,$$

$$\Psi^\dagger_\gamma(\vec{r},t) = \frac{1}{\sqrt{A}}\sum_{\vec{q}} e^{iE_q t/\hbar - i\vec{q}\cdot\vec{r}} c^\dagger_q,$$

$$E_q = \hbar c q.$$

The massive α particle can decay to an α and a β particle via the interaction

$$H_{\rm int} = g\int dxdy\,\left[\Phi_\alpha(\vec{r},t)\Psi^\dagger_\beta(\vec{r},t)\Psi^\dagger_\gamma(\vec{r},t) + \Phi^\dagger_\alpha(\vec{r},t)\Psi_\beta(\vec{r},t)\Psi_\gamma(\vec{r},t)\right],$$

where the coupling constant g is small. The creation and destruction operators obey the commutation rules $[a_{\vec{k}}, a_{\vec{k}'}] = \delta_{\vec{k}\vec{k}'}$, $[b_{\vec{q}}, b_{\vec{q}'}] = \delta_{\vec{q}\vec{q}'}$ and $[c_{\vec{q}}, c_{\vec{q}'}] = \delta_{\vec{q}\vec{q}'}$. FYI: In two dimensions $\delta_{\vec{k}\vec{k}'} = \delta_{k_x k'_x}\delta_{k_y k'_y}$.

(a) (5 pts) Evaluate the commutator $[\Phi_\alpha(\vec{r},t), \Phi^\dagger_\alpha(\vec{r}',t)]$.

(b) (5 pts) What is the dimensionality of g?

(c) (20 pts) Calculate the rate at which an α particle at rest decays into a β and a γ particle.

(d) (5 pts) How would your answer change if the β and γ particles were identical?

(Extra work space for #1)

a) $\left[\Phi(\vec r, t), \Phi^+(\vec r\,', t)\right]$

$$= \frac{1}{A}\sum_{\vec k \vec k'} e^{-iE_k t/\hbar + i\vec k\cdot\vec r}\, e^{iE_{k'} t/\hbar - i\vec k'\cdot\vec r\,'}\left[a_k, a^+_{k'}\right]$$

$$= \frac{1}{A}\sum_{k} e^{i\vec k\cdot(\vec r - \vec r\,')} = \frac{A}{A(2\pi)^2}\int dk_x\, dk_y\, e^{ik_x(x-x')}\cdot e^{ik_y(y-y')}$$

$$= \delta(x-x')\,\delta(y-y') = \delta(\vec r - \vec r\,')$$

b) $[E] = [g]\cdot[L^2]\cdot[\Phi\, \Psi^2]$

dimension of Φ or $\Psi = \left[\frac{1}{L}\right]$

from commutation relation

$$[E] = [g]\left[\frac{1}{L}\right]$$

$$[g] = \text{energy}\cdot\text{length}$$

c) $$\Gamma = \sum_{qq'} \frac{2\pi}{\hbar}\left|M_{qq'}\right|^2 \delta(E_k - E_q - E_{q'})$$

$$M_{qq'} = \langle 0 | a_q a_{q'} H_{int}\, a^+_{k=0} | 0\rangle$$

$$= \frac{g}{A^{3/2}}\int d^2r\, e^{-i\vec q\cdot\vec r - i\vec q\,'\cdot\vec r}$$

$$= \frac{g}{A^{1/2}}\,\delta_{q,-q'}$$

$$\left|M_{qq'}\right|^2 = \frac{g^2}{A}\,\delta_{q,-q'}$$

(Extra work space for #1)

$$\Gamma = \frac{2\pi}{\hbar}\frac{g^2}{A}\sum_q \delta(M_A c^2 - 2\hbar c q)$$

$$= \frac{2\pi}{\hbar}\frac{g^2}{A}\cdot\frac{A}{(2\pi)^2}\int 2\pi q\, dq\, \delta(M_a c^2 - 2\hbar c q)$$

$$= \frac{g^2}{\hbar}\int q\, dq\, \delta(M_a c^2 - 2\hbar c q)$$

$$= \frac{g^2}{2\hbar^2 c} q \quad , \quad q = \frac{M_a c^2}{2\hbar c} = \frac{M_a c}{2\hbar}$$

$$= \frac{g^2}{2\hbar^2 c}\frac{M_a c}{2\hbar} = \boxed{\frac{g^2 M_a}{4\hbar^3}}$$

check units

$$\left[\frac{1}{t}\right] \stackrel{?}{=} [E^2][L^2]\frac{[M]}{[E^3][t^3]} = \frac{1}{[t]}\frac{1}{[E]}\left[\frac{ML^2}{t^2}\right]$$

$\uparrow = [E]$

$$\stackrel{?}{=} \frac{1}{t} \quad \checkmark$$

d) Matrix element would double so $|M|^2$ increases by 4.

But → only sum over $\frac{1}{2}$ as many q values ($q, -\vec{q}$) are same state. So overall

Γ would double

2. Imagine you had calculated the following matrix element,

$$\boldsymbol{\mathcal{M}} = \langle \alpha, J_f = 1, M_f = 0 | z^2 | \beta, J_i = 3, M_i = 0 \rangle.$$

For the following matrix elements, first state whether they are zero, and if not, express them in terms of $\boldsymbol{\mathcal{M}}$. **Evaluate any Clebsch-Gordan coefficients in your answers. You can use tables or online resources to get the values.** (5 pts. each)

- $\langle \alpha, J_f = 1, M_f = 0 | x^2 + y^2 | \beta, J_i = 3, M_i = 0 \rangle$

I know $\langle J=1 | x^2+y^2+z^2 | J=3 \rangle = 0$

so

$\langle \alpha | J_f = 1, M_f = 0 | x^2 y^2 | \beta, J_i = 3, M_i = 0 \rangle = -\mathcal{M}$

- $\langle \alpha, J_f = 1, M_f = 0 | z^2 | \beta, J_i = 3, M_i = 2 \rangle$

$z^2 \sim T^k_{q=0}$ so

$= 0$

- $\langle \alpha, J_f = 1, M_f = 0 | xz | \beta, J_i = 3, M_i = 0 \rangle$

$xz \sim T^2_1$ or T^2_{-1}, so

$= 0$

- $\langle \alpha, J_f = 1, M_f = 0 | xz | \beta, J_i = 3, M_i = 2 \rangle$

same reasons

$= 0$

- $\langle \alpha, J_f = 1, M_f = 0 | xy | \beta, J_i = 3, M_i = 2 \rangle$

$\mathcal{M} = \langle M_f = 0 | z^2 | M_i = 0 \rangle = \langle 0 | \frac{2}{3} T^2_0 + \frac{1}{3} T^0_0 r^2 | 0 \rangle$

$\langle m_f = 0 | xy | M_i = 2 \rangle = \langle 0 | \frac{i}{\sqrt{6}} (T^2_{-2} - T^2_2) | 0 \rangle$

$= \frac{i}{\sqrt{6}} \frac{\langle 1,0 | 2,-2,3,2 \rangle}{\langle 1 0 | 2,0,3,0 \rangle} \cdot \frac{3}{2} \mathcal{M}$

$= \frac{3i}{2\sqrt{6}} \mathcal{M} \frac{1/\sqrt{7}}{3/\sqrt{35}} = i \frac{\sqrt{5}}{2\sqrt{6}} \mathcal{M}$

(Extra work space for #2)

3. Electrons of mass m are placed in a three-dimensional spherically symmetric harmonic oscillator with potential, $V = m\omega^2 r^2/2$.

 (a) (5 pts) What is the ground state energy for a single electron?

$\frac{3}{2}\hbar\omega$

 (b) (5 pts) What is the net ground state energy if 20 electrons are in the well?

$N = 0 \to 2$ electrons $\quad \epsilon = 3/2\,\hbar\omega \quad 3\hbar\omega$

$N = 1\ (\ell=1) \to 6$ electrons $\quad \epsilon = 5/2\,\hbar\omega \quad 15\hbar\omega$

$N = 2\ (\ell=2) \to 10$ electrons $\quad \epsilon = 7/2\,\hbar\omega \quad 35\hbar\omega$

$N = 2\ (\ell=0) \to 2$ electrons $\quad \epsilon = 7/2\,\hbar\omega \quad 7\hbar\omega$

20 $\quad$ $60\hbar\omega$

Now add an interaction with an external magnetic field,

$$V_\beta = -\mu\vec{B}\cdot(\vec{L} + 2\vec{S}),$$

where the strength of the magnetic field is adjusted so that $\mu B = \omega$.

 (c) (10 pts) What is the new single-particle ground state energy?

$E = \frac{1}{2}\hbar\omega$ for $N=0,\ m_s = 1/2,\ \ell = 0$

or any other $N,\ \ell = N,\ m_s = 1/2,\ m_\ell = \ell$

$E = (N+\frac{3}{2})\hbar\omega - \mu B\hbar(1+N) = \frac{1}{2}\hbar\omega$

 (d) (10 pts) What is the net ground state energy if 20 electrons are in the well?

Each level if $E = \frac{1}{2}\hbar\omega$ can hold one electron, so you can put all 20 electrons in different levels with that energy.

$E_{tot} = 2\cdot\frac{1}{2}\hbar\omega = 10\hbar\omega$

(Extra work space for #3)