PHY 851 – QUANTUM MECHANICS Your Name: ______________________________
MIDTERM II, October 24, 2021

$$\int_{-\infty}^{\infty} dx\ e^{-x^2/(2a^2)} = a\sqrt{2\pi},$$

$$H = i\hbar\partial_t,\ \vec{P} = -i\hbar\nabla,$$

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

$$U(t,-\infty) = 1 + \frac{-i}{\hbar}\int_{-\infty}^{t} dt'\ V(t')U(t',-\infty),$$

$$\langle x|x'\rangle = \delta(x-x'),\ \langle p|p'\rangle = \frac{1}{2\pi\hbar}\delta(p-p'),$$

$$|p\rangle = \int dx\ |x\rangle e^{ipx/\hbar},\quad |x\rangle = \int \frac{dp}{2\pi\hbar}|p\rangle e^{-ipx/\hbar},$$

$$H = \frac{P^2}{2m} + \frac{1}{2}m\omega^2x^2 = \hbar\omega(a^\dagger a + 1/2),$$

$$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}X - i\sqrt{\frac{1}{2\hbar m\omega}}P,$$

$$\psi_0(x) = \frac{1}{(\pi b^2)^{1/4}}e^{-x^2/2b^2},\quad b^2 = \frac{\hbar}{m\omega},$$

$$\rho(\vec{r},t) = \psi^*(\vec{r}_1,t_1)\psi(\vec{r}_2,t_2)$$

$$\vec{j}(\vec{r},t) = \frac{-i\hbar}{2m}(\psi^*(\vec{r},t)\nabla\psi(\vec{r},t) - (\nabla\psi^*(\vec{r},t))\psi(\vec{r},t)) - \frac{e\vec{A}}{mc}|\psi(\vec{r},t)|^2.$$

$$H = \frac{(\vec{P} - e\vec{A}/c)^2}{2m} + e\Phi,$$

$$\text{For } V = \beta\delta(x-y):\quad -\frac{\hbar^2}{2m}\left(\frac{\partial}{\partial x}\psi(x)|_{y+\epsilon} - \frac{\partial}{\partial x}\psi(x)|_{y-\epsilon}\right) = -\beta\psi(y),$$

$$\vec{E} = -\nabla\Phi - \frac{1}{c}\partial_t\vec{A},\quad \vec{B} = \nabla\times\vec{A},$$

$$\omega_{\rm cyclotron} = \frac{eB}{mc},$$

$$e^{A+B} = e^Ae^Be^{-C/2},\ \text{ if } [A,B] = C,\ \text{ and } [C,A] = [C,B] = 0,$$

$$Y_{0,0} = \frac{1}{\sqrt{4\pi}},\quad Y_{1,0} = \sqrt{\frac{3}{4\pi}}\cos\theta,\quad Y_{1,\pm1} = \mp\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\pm\phi},$$

$$Y_{2,0} = \sqrt{\frac{5}{16\pi}}(3\cos^2\theta - 1),\quad Y_{2,\pm1} = \mp\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta e^{\pm i\phi},$$

$$Y_{2,\pm2} = \sqrt{\frac{15}{32\pi}}\sin^2\theta e^{\pm2i\phi},\quad Y_{\ell-m}(\theta,\phi) = (-1)^mY^*_{\ell m}(\theta,\phi).$$

$$|N\rangle = |n\rangle - \sum_{m\neq n} |m\rangle \frac{1}{\epsilon_m - \epsilon_n} \langle m|V|n\rangle + \cdots$$

$$E_N = \epsilon_n + \langle n|V|n\rangle - \sum_{m\neq n} \frac{|\langle m|V|n\rangle|^2}{\epsilon_m - \epsilon_n}$$

$$j_0(x) = \frac{\sin x}{x},\ n_0(x) = -\frac{\cos x}{x},\ j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x},\ n_1(x) = -\frac{\cos x}{x^2} - \frac{\sin x}{x}$$

$$j_2(x) = \left(\frac{3}{x^3} - \frac{1}{x}\right)\sin x - \frac{3}{x^2}\cos x,\ n_2(x) = -\left(\frac{3}{x^3} - \frac{1}{x}\right)\cos x - \frac{3}{x^2}\sin x,$$

$$\frac{d}{dt}P_{i\to n}(t) = \frac{2\pi}{\hbar}|V_{ni}|^2\delta(E_n - E_i),$$

$$\frac{d\sigma}{d\Omega} = \frac{m^2}{4\pi^2\hbar^4}\left|\int d^3r\mathcal{V}(r)e^{i(\vec{k}_f - \vec{k}_i)\cdot\vec{r}}\right|^2,$$

$$\sigma = \frac{(2S_R+1)}{(2S_1+1)(2S_2+1)}\frac{4\pi}{k^2}\frac{(\hbar\Gamma_R/2)^2}{(\epsilon_k - \epsilon_r)^2 + (\hbar\Gamma_R/2)^2},$$

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_{\rm single}\tilde{S}(\vec{q}),\quad \tilde{S}(\vec{q}) = \frac{1}{N}\left|\sum_{\vec{a}} e^{i\vec{q}\cdot\vec{a}}\right|^2,$$

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_{\rm Rutherford} e^{i\vec{k}\cdot\vec{r}} \qquad = \sum_{\ell}(2\ell$$

$$P_\ell(\cos\theta) = \sqrt{\frac{4\pi}{2\ell+1}}Y_{\ell,m=0}(\theta,\phi),$$

$$P_0(x) = 1,\ P_1(x) = x,\ P_2(x) = (3x^2-1)/3,$$

$$f(\Omega) \equiv \sum_\ell (2\ell+1)e^{i\delta_\ell}\sin\delta_\ell \frac{1}{k}P_\ell(\cos\theta)$$

$$\psi_{\vec{k}}(\vec{r})|_{R\to\infty} = e^{i\vec{k}\cdot\vec{r}} + \frac{e^{ikr}}{r}f(\Omega),$$

$$\frac{d\sigma}{d\Omega} = |f(\Omega)|^2,\quad \sigma = \frac{4\pi}{k^2}\sum_\ell (2\ell+1)\sin^2\delta_\ell,\quad \delta \approx -ak$$

$$L_\pm|\ell,m\rangle = \sqrt{\ell(\ell+1) - m(m\pm1)}|\ell,m\pm1\rangle,$$

$$C^{\ell,s}_{m_\ell,m_s;JM} = \langle \ell,s,J,M|\ell,s,m_\ell,m_s\rangle,$$

$$\langle\tilde{\beta},J,M|T^k_q|\beta,\ell,m_\ell\rangle = C^{k\ell}_{qm_\ell;JM}\frac{\langle\tilde{\beta},J||T^{(k)}||\beta,\ell,J\rangle}{\sqrt{2J+1}},$$

$$n = \frac{(2s+1)}{(2\pi)^d}\int_{k<k_f} d^dk,\quad d\ \text{dimensions},$$

$$\{\Psi_s(\vec{x}),\Psi^\dagger_{s'}(\vec{y})\} = \delta^3(\vec{x}-\vec{y})\delta_{ss'},$$

$$\Psi^\dagger_s(\vec{r}) = \frac{1}{\sqrt{V}}\sum_{\vec{k}} e^{i\vec{k}\cdot\vec{r}}a^\dagger_s(\vec{k}),\quad \{\Psi_s(\vec{x}),a^\dagger_\alpha\} = \phi_{\alpha,s}(\vec{x}).$$

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1. (15 pts) At $t = 0$ an electron is in the $|\uparrow\rangle$ (up along the z axis) state, which is represented by

$$|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

The evolution is determined by the Hamiltonian,

$$H = A\sigma_z + B\sigma_y.$$

What is the probability the electron will be found in the $|\downarrow\rangle$ state as a function of time?

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Solution:

$$
\begin{aligned}
e^{-iHt/\hbar} &= e^{-i\sqrt{A^2+B^2}\vec{\sigma}\cdot\hat{n}t/\hbar} \\
\vec{\sigma}\cdot\hat{n} &= \frac{1}{\sqrt{A^2+B^2}}(A\sigma_z + B\sigma_y), \\
e^{-iHt/\hbar} &= \cos(\sqrt{A^2+B^2}t/\hbar) - i\vec{\sigma}\cdot\hat{n}\sin(\sqrt{A^2+B^2}t/\hbar), \\
\langle\downarrow|e^{-iHt/\hbar}|\uparrow\rangle &= \frac{iB}{\sqrt{A^2+B^2}}\sin(\sqrt{A^2+B^2}t/\hbar), \\
\text{Prob} &= \frac{B^2}{A^2+B^2}\sin^2(\sqrt{A^2+B^2}t/\hbar).
\end{aligned}
$$

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2. Consider a **TWO-DIMENSIONAL** world with two types of particles, an *Aaron* particle and a *Barbara* particle. The Aaron particle in state a can decay into a Barbara particle in state b via the interaction,

$$\langle \text{Barbara}, b|V|\text{Aaron}, a\rangle = \int dxdy\ \psi_b^*(x,y)v_0\psi_a(x,y).$$

The Aaron and Barbara particles have the same mass m, but Aaron particles feel a harmonic oscillator potential,

$$V_A(x,y) = \frac{1}{2}m\omega^2(x^2+y^2),$$

while the Barbara particles feel no such potential. An Aaron particle in the ground state of the harmonic oscillator decays into a Barbara particle. Assume that v_0 is sufficiently small that Fermi's golden rule can be applied.

(a) (5 pts) What is the magnitude of k, the outgoing momentum wave vector of the Barbara particle?

(b) (15 pts) What is the matrix element $\langle \text{Barbara}, b|V|\text{Aaron}, a\rangle$? Barbara's state b is a plane wave of wave number $\vec{k}$,

$$\psi_b = \frac{e^{i\vec{k}\cdot\vec{r}}}{\sqrt{A}}, \quad A = \text{area} \to \infty.$$

Aaron particle's state a refers to the ground state of the harmonic oscillator,

$$\psi_a = \frac{1}{\pi^{1/2}b}e^{-(x^2+y^2)/2b^2}, \quad b = \sqrt{\frac{\hbar}{m\omega}}.$$

(c) (25 pts) What is the decay rate of an Aaron? (You will be penalized if your answer is dimensionally inconsistent)

Solution:

a)

$$\begin{aligned} E_{\text{Barbara}} &= \hbar\omega, \\ k &= \sqrt{\frac{2mE_{\text{Barbara}}}{\hbar^2}} \\ &= \sqrt{\frac{2m\hbar\omega}{\hbar^2}}. \end{aligned}$$

b) Let's choose k in the x direction. (answer can't depend on direction)

$$\begin{aligned} \langle \text{Barbara}, b|V|\text{Aaron}, a\rangle &= v_0 \int dxdy\ \frac{e^{-ikx}}{\sqrt{A}} \frac{1}{\pi^{1/2}b} e^{-(x^2+y^2)/2b^2}, \\ &= \frac{v_0}{bA^{1/2}\pi^{1/2}} \int dxdy\ e^{-(x+ikb^2)^2/2b^2 - y^2/2b^2} e^{-k^2b^2/2} \\ &= \frac{2b\pi^{1/2}}{A^{1/2}} v_0 e^{-k^2b^2/2}. \end{aligned}$$

c)

$$\begin{aligned} \Gamma &= \frac{2\pi}{\hbar} \sum_{k_x k_y} |\langle \text{Barbara}, b|V|\text{Aaron}, a\rangle|^2 \delta(E_{\text{Barbara}} - \hbar\omega), \\ &= \frac{2\pi}{\hbar} \frac{A}{(2\pi)^2} \int 2\pi k dk\ |\langle \text{Barbara}, b|V|\text{Aaron}, a\rangle|^2 \delta(E_{\text{Barbara}} - \hbar\omega) \\ &= \frac{k}{\hbar} A \frac{|\langle \text{Barbara}, b|V|\text{Aaron}, a\rangle|^2}{\hbar^2 k/m}, \\ &= \frac{4\pi v_0^2}{\hbar} \frac{kb^2}{\hbar^2 k/m} e^{-k^2b^2} \\ &= \frac{4\pi m v_0^2 b^2}{\hbar^3} e^{-k^2b^2}. \end{aligned}$$

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3. A point charge Ze is placed at a point $\vec{r} = 0$, and the differential cross section is measured with a beam of electrons of wave number k moving along the z axis. The Rutherford differential cross section is

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{point}} = \alpha = \frac{m^2 Z^2 e^4}{k^4(1-\cos\theta)^2}.$$

Now, that same charge Ze is spread out uniformly along a line from $z = -a$ to $z = a$. I.e. the charge density is

$$\rho(x,y,z) = \begin{cases} 0, & z < -a \\ \delta(x)\delta(y)\frac{Ze}{2a}, & -a < z < a \\ 0, & a < z \end{cases}.$$

The cross section is measured again.

(a) (15 pts) What is the differential cross section? (Express answer in terms of α, k, and a)

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{line}} = ???$$

(b) (5 pts) What is $(d\sigma/d\Omega)_{\text{line}}$ in the limit that $a \to 0$?

(c) (10 pts) At what scattering angles does the differential cross section disappear?

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Solution:
a) The form factor goes as

$$\begin{aligned} F(\Omega) &= \frac{1}{Ze}\int d^3r\ \rho(\vec{r})e^{i\vec{q}\cdot\vec{r}} \\ &= \frac{1}{2a}\int_{-a}^{a} dz\ e^{iq_z z} \\ &= \frac{1}{2q_z a}(e^{iq_z a} - e^{-iq_z a}) \\ &= i\frac{\sin(q_z a)}{q_z a}, \\ \left(\frac{d\sigma}{d\Omega}\right)_{\text{line}} &= \left(\frac{d\sigma}{d\Omega}\right)_{\text{point}} \left(\frac{\sin[ka(1-\cos\theta)]}{ka(1-\cos\theta)}\right)^2 \\ &= \alpha \left(\frac{\sin[ka(1-\cos\theta)]}{ka(1-\cos\theta)}\right)^2. \end{aligned}$$

b) As $ka \to 0$, $\sin(kax)/kax = 1$,

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{line}} \to \alpha.$$

c)

$$\begin{aligned} \sin[ka(1-\cos\theta)] &= 0, \\ ka(1-\cos\theta_n) &= n\pi, \\ \cos\theta_n &= 1 - \frac{n\pi}{ka}. \end{aligned}$$