PHY 851 – QUANTUM MECHANICS
MIDTERM II
October 23-25, 2020

- Submit your exam via GRADESCOPE. Be sure that no page contains responses for more than one problem. Having different parts 2b,2c$\cdots$ of a given problem on a single page is fine. Individual problems can span multiple pages.
- Once the exam is opened, you are to upload your answer within six hours.
- **Do not write your name on the exam**
- This exam is open-book, open-notes, open-internet but closed-mouth. You are permitted to use mathematical software, e.g. Mathematica. You are not to communicate with any other individuals regarding the exam, during the exam.

$$\int_{-\infty}^{\infty} dx\; e^{-x^2/(2a^2)} = a\sqrt{2\pi},$$

$$H = i\hbar\partial_t,\;\; \vec{P} = -i\hbar\nabla,$$

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

$$U(t,-\infty) = 1 + \frac{-i}{\hbar}\int_{-\infty}^{t} dt'\; V(t')U(t',-\infty),$$

$$\langle x|x'\rangle = \delta(x-x'),\; \langle p|p'\rangle = \frac{1}{2\pi\hbar}\delta(p-p'),$$

$$|p\rangle = \int dx\; |x\rangle e^{ipx/\hbar}, \quad |x\rangle = \int \frac{dp}{2\pi\hbar}|p\rangle e^{-ipx/\hbar},$$

$$H = \frac{P^2}{2m} + \frac{1}{2}m\omega^2 x^2 = \hbar\omega(a^\dagger a + 1/2),$$

$$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}X - i\sqrt{\frac{1}{2\hbar m\omega}}P,$$

$$\psi_0(x) = \frac{1}{(\pi b^2)^{1/4}}e^{-x^2/2b^2}, \quad b^2 = \frac{\hbar}{m\omega},$$

$$\rho(\vec{r},t) = \psi^*(\vec{r}_1,t_1)\psi(\vec{r}_2,t_2)$$

$$\vec{j}(\vec{r},t) = \frac{-i\hbar}{2m}(\psi^*(\vec{r},t)\nabla\psi(\vec{r},t) - (\nabla\psi^*(\vec{r},t))\psi(\vec{r},t)) - \frac{e\vec{A}}{mc}|\psi(\vec{r},t)|^2.$$

$$H = \frac{(\vec{P} - e\vec{A}/c)^2}{2m} + e\Phi,$$

$$\text{For } V = \beta\delta(x-y): \quad -\frac{\hbar^2}{2m}\left(\frac{\partial}{\partial x}\psi(x)|_{y+\epsilon} - \frac{\partial}{\partial x}\psi(x)|_{y-\epsilon}\right) = -\beta\psi(y),$$

$$\vec{E} = -\nabla\Phi - \frac{1}{c}\partial_t\vec{A}, \quad \vec{B} = \nabla\times\vec{A},$$

$$\omega_{\text{cyclotron}} = \frac{eB}{mc},$$

$$e^{A+B} = e^A e^B e^{-C/2}, \;\; \text{if } [A,B] = C, \text{ and } [C,A] = [C,B] = 0,$$

$$Y_{0,0} = \frac{1}{\sqrt{4\pi}}, \quad Y_{1,0} = \sqrt{\frac{3}{4\pi}}\cos\theta, \quad Y_{1,\pm1} = \mp\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\pm\phi},$$

$$Y_{2,0} = \sqrt{\frac{5}{16\pi}}(3\cos^2\theta - 1), \quad Y_{2,\pm1} = \mp\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta e^{\pm i\phi},$$

$$Y_{2,\pm2} = \sqrt{\frac{15}{32\pi}}\sin^2\theta e^{\pm 2i\phi}, \quad Y_{\ell -m}(\theta,\phi) = (-1)^m Y^*_{\ell m}(\theta,\phi).$$

$$|N\rangle = |n\rangle - \sum_{m\neq n} |m\rangle \frac{1}{\epsilon_m - \epsilon_n} \langle m|V|n\rangle + \cdots$$

$$E_N = \epsilon_n + \langle n|V|n\rangle - \sum_{m\neq n} \frac{|\langle m|V|n\rangle|^2}{\epsilon_m - \epsilon_n}$$

$$j_0(x) = \frac{\sin x}{x},\ n_0(x) = -\frac{\cos x}{x},\ j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x},\ n_1(x) = -\frac{\cos x}{x^2} - \frac{\sin x}{x}$$

$$j_2(x) = \left(\frac{3}{x^3} - \frac{1}{x}\right)\sin x - \frac{3}{x^2}\cos x,\ n_2(x) = -\left(\frac{3}{x^3} - \frac{1}{x}\right)\cos x - \frac{3}{x^2}\sin x,$$

$$\frac{d}{dt}P_{i\to n}(t) = \frac{2\pi}{\hbar}|V_{ni}|^2\delta(E_n - E_i),$$

$$\frac{d\sigma}{d\Omega} = \frac{m^2}{4\pi^2\hbar^4}\left|\int d^3r\mathcal{V}(r)e^{i(\vec{k}_f - \vec{k}_i)\cdot\vec{r}}\right|^2,$$

$$\sigma = \frac{(2S_R+1)}{(2S_1+1)(2S_2+1)}\frac{4\pi}{k^2}\frac{(\hbar\Gamma_R/2)^2}{(\epsilon_k - \epsilon_r)^2 + (\hbar\Gamma_R/2)^2},$$

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_{\text{single}}\tilde{S}(\vec{q}),\quad \tilde{S}(\vec{q}) = \left|\sum_{\vec{a}} e^{i\vec{q}\cdot\vec{a}}\right|^2,$$

$$e^{i\vec{k}\cdot\vec{r}} = \sum_\ell (2\ell+1)i^\ell j_\ell(kr)P_\ell(\cos\theta),$$

$$P_\ell(\cos\theta) = \sqrt{\frac{4\pi}{2\ell+1}}Y_{\ell,m=0}(\theta,\phi),$$

$$P_0(x) = 1,\ P_1(x) = x,\ P_2(x) = (3x^2-1)/3,$$

$$f(\Omega) \equiv \sum_\ell (2\ell+1)e^{i\delta_\ell}\sin\delta_\ell \frac{1}{k}P_\ell(\cos\theta)$$

$$\psi_{\vec{k}}(\vec{r})|_{R\to\infty} = e^{i\vec{k}\cdot\vec{r}} + \frac{e^{ikr}}{r}f(\Omega),$$

$$\frac{d\sigma}{d\Omega} = |f(\Omega)|^2,\quad \sigma = \frac{4\pi}{k^2}\sum_\ell (2\ell+1)\sin^2\delta_\ell,\quad \delta \approx -ak$$

$$L_\pm|\ell,m\rangle = \sqrt{\ell(\ell+1) - m(m\pm1)}|\ell,m\pm1\rangle,$$

$$C^{\ell,s}_{m_\ell,m_s;JM} = \langle \ell,s,J,M|\ell,s,m_\ell,m_s\rangle,$$

$$\langle\tilde{\beta},J,M|T^k_q|\beta,\ell,m_\ell\rangle = C^{k\ell}_{qm_\ell;JM}\frac{\langle\tilde{\beta},J||T^{(k)}||\beta,\ell,J\rangle}{\sqrt{2J+1}},$$

$$n = \frac{(2s+1)}{(2\pi)^d}\int_{k<k_f} d^dk,\quad d\text{ dimensions},$$

$$\{\Psi_s(\vec{x}),\Psi^\dagger_{s'}(\vec{y})\} = \delta^3(\vec{x}-\vec{y})\delta_{ss'},$$

$$\Psi^\dagger_s(\vec{r}) = \frac{1}{\sqrt{V}}\sum_{\vec{k}} e^{i\vec{k}\cdot\vec{r}}a^\dagger_s(\vec{k}),\quad \{\Psi_s(\vec{x}),a^\dagger_\alpha\} = \phi_{\alpha,s}(\vec{x}).$$

1. A particle of mass m and charge e feels a strong constant electric field $\vec{E} = E_0\hat{x}$. Additionally, the particle experiences a constant weak magnetic field $\vec{B} = B_0\hat{z}$. The magnetic field strength is less than the electric field strength, $E_0 > B_0$.

   (a) (10 pts) Describe a reference frame where either $\vec{E}$ or $\vec{B}$ is zero. State the velocity (both magnitude and direction) of that frame, and describe the fields in that frame.

   (b) Averaged over a sufficiently long time interval Δ to avoid any oscillatory terms, answer these questions about the components of the average velocity $(\bar{v}_x(t), \bar{v}_y(t), \bar{v}_z(t))$ at large times, $t >> \Delta t$. Assume the particle might has some original momentum with non-zero components in each direction. Circle the correct answers:

      i. (5 pts) Which components average to zero? $[v_x, \; v_y, \; v_z]$

      ii. (5 pts) Which components average to a constant? (but not necessarily zero) $[v_x, \; v_y, \; v_z]$

      iii. (5 pts) Which components continue to accelerate over time? $[v_x, \; v_y, \; v_z]$

   You might choose a gauge where the vector potential $\vec{A}$ is pointed along the ~~x~~ y axis.

(a) $\vec{A} = B_0 x\,\hat{y}$, $A_0 = -E_0 x$,

$$\vec{v}/c = \frac{B_0}{E_0}\hat{y}$$

$$A_y' = \gamma\left(A_y + \frac{v}{c}A_0\right) = 0$$

$$A_0' = \gamma\left(A_0 + \frac{v}{c}A_y\right) = -\gamma\left(E_0 - \frac{B_0^2}{E_0}\right)x$$

$$E_x' = \gamma\left(E_0 - \frac{B_0^2}{E_0}\right) = \frac{E_0}{\gamma}$$

in this frame, the particle accelerates uniformly in x-direction

in lab frame, you add velocity $\frac{B_0}{E_0}\hat{y}$

i) V_x

ii) V_x, V_y, V_z

iii) none

2. Consider a particle of mass m_0 confined to a two-dimensional harmonic oscillator,

$$V(x, y) = \frac{1}{2} m\omega^2 (r^2), \quad r = \sqrt{x^2 + y^2}.$$

Consider solutions of the form,

$$\psi_{nm}(x, y) = \phi_{nm}(r) e^{im\phi}, \quad m = \text{ an integer.}$$

(a) (10 pts) Write a one-dimensional differential equation (Schrödinger's equation) in terms of r for $\phi_{nm}(r)$ assuming ϕ_{nm} is an eigenstate with energy E_{nm}. Here, n denotes different eigenstates with the same m.

(b) (10 pts) What are the eigen-energies? Also state the degeneracies of each level. (easier to work out in Cartesian basis)

(c) (5 pts) Consider the state (un-normalized)

$$\psi_{NM} = (a_x^\dagger + i a_y^\dagger)^M (a_x^\dagger a_x^\dagger + a_y^\dagger a_y^\dagger)^{(N-M)/2} |0\rangle, \quad M \geq 0$$
$$\psi_{NM} = (a_x^\dagger - i a_y^\dagger)^{|M|} (a_x^\dagger a_x^\dagger + a_y^\dagger a_y^\dagger)^{(N-M)/2} |0\rangle, \quad M < 0$$

with energy $(N + 1)\hbar\omega$. Here, $a_x^\dagger$ and $a_y^\dagger$ are the raising operators in the Cartesian basis. Show that under rotations that ψ_{NM} behaves as $e^{iM\phi}$. I.e., show that

$$R(\Delta\phi)\psi_{NM}(r, \phi) = e^{iM\Delta\phi}\psi_{NM}(r, \phi).$$

You may use the fact that $a_x^\dagger$ and $a_y^\dagger$ rotate amongst each other like x and y.

(d) (5 pts) For a given eigen-energy $(N + 1)\hbar\omega$, what values of M are possible?

a) $\left[-\frac{\hbar^2}{2m}\left(\partial_r^2 + \frac{1}{r} + \frac{1}{r^2}\partial_\varphi^2\right) + V(r)\right]\psi = E\psi$

$\left\{-\frac{\hbar^2}{2m}\left(\partial_r^2 + \frac{1}{r}\partial_r\right) + \frac{1}{2}m\omega^2 r^2 + \frac{\hbar^2 m^2}{2mr^2}\right\}\varphi_{nm} = E\varphi_{nm}$

b) $E_n = (n_x + n_y + 1)\hbar\omega = (N+1)\hbar\omega, \; N = 0, 1, 2, 3\ldots$

degen $(N) = N + 1$

c) $(a_x^\dagger + i a_y^\dagger) \sim (a_x^\dagger \cos\varphi + a_y^\dagger \sin\varphi + i(a_y^\dagger \cos\varphi - a_x^\dagger \sin\varphi))$

$= (a_x^\dagger + i a_y^\dagger) e^{-i\varphi}$

$(a_x^\dagger + i a_y^\dagger)^M \sim (a_x^\dagger + i a_y^\dagger)^M e^{iM\varphi}$

d) To match degeneracy, parity

$M = -N, -N+2, -N+4 \cdots N-2, N$

or by inspecting form of ψ_{NM}

3. (20 pts) Two distinguishable spinless particles of mass $\boldsymbol{m}$ occupy the same $\boldsymbol{\ell = 1}$ eigenstate of a spherically symmetric potential. An spin-spin interaction is then applied,

$$\boldsymbol{V}_{\text{ss}} = -\beta \vec{\boldsymbol{L}}_1 \cdot \vec{\boldsymbol{L}}_2.$$

(a) (5 pts) What values of the total angular momentum might the system attain?

(b) (15 pts) What are the shifts of the energy due to $\boldsymbol{V_{ss}}$? Assume you needn't worry about mixing with other levels.

a) $0, 1, 2 = j$

b) $\vec{J} = \vec{L}_1 + \vec{L}_2$, $\quad \vec{L}_1 \cdot \vec{L}_2 = \frac{1}{2}\left[|\vec{J}|^2 - |\vec{L}_1|^2 - |\vec{L}_2|^2\right]$

$$= \frac{1}{2}\left[j(j+1) - 2\ell(\ell+1)\right]$$

$\uparrow_{\ell = 1}$

$$\Delta E = \begin{cases} -\beta, & j = 2 \\ \beta, & j = 1 \\ 2\beta, & j = 0 \end{cases}$$

4. (20 pts) Two spinless particles ("1" and "2") of mass m occupy an $\ell = 1$ eigenstate and an $\ell = 2$ eigenstate of a spherically symmetric potential. A magnetic field with the following interaction is then applied,

$$V_b = -\mu_1 \vec{B} \cdot \vec{L}_1 - \mu_2 \vec{B} \cdot \vec{L}_2.$$

(a) (5 pts) What values of the total angular momentum might the system attain?

(b) (15 pts) What are the shifts of the energy due to V_b? Assume you needn't worry about mixing with other levels.

a) $J = 1, 2, 3$

b) $\Delta E = -B\hbar \{\mu_1 m_1 + \mu_2 m_2\}$

ranges from $-\mu_1 B\hbar - 2\mu_2 B\hbar$

to $+\mu_1 B\hbar + 2\mu_2 B\hbar$