PHY 851 – QUANTUM MECHANICS
MIDTERM I

September 25-28, 2020

- Submit your exam via GRADESCOPE. Be sure that no page contains responses for more than one problem. Having different parts 2b,2c$\cdots$ of a given problem on a single page is fine. Individual problems can span multiple pages.
- Once the exam is opened, you are to upload your answer within six hours.
- This exam is open-book, open-notes, open-internet but closed-mouth. You are permitted to use mathematical software, e.g. Mathematica. You are not to communicate with any other individuals regarding the exam, during the exam.

$$\int_{-\infty}^{\infty} dx\ e^{-x^2/(2a^2)} = a\sqrt{2\pi},$$

$$H = i\hbar\partial_t,\ \vec{P} = -i\hbar\nabla,$$

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

$$U(t,-\infty) = 1 + \frac{-i}{\hbar}\int_{-\infty}^{t} dt'\ V(t')U(t',-\infty),$$

$$\langle x|x'\rangle = \delta(x-x'),\ \langle p|p'\rangle = \frac{1}{2\pi\hbar}\delta(p-p'),$$

$$|p\rangle = \int dx\ |x\rangle e^{ipx/\hbar}, \quad |x\rangle = \int \frac{dp}{2\pi\hbar}|p\rangle e^{-ipx/\hbar},$$

$$H = \frac{P^2}{2m} + \frac{1}{2}m\omega^2 x^2 = \hbar\omega(a^\dagger a + 1/2),$$

$$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}X - i\sqrt{\frac{1}{2\hbar m\omega}}P,$$

$$\psi_0(x) = \frac{1}{(\pi b^2)^{1/4}}e^{-x^2/2b^2}, \quad b^2 = \frac{\hbar}{m\omega},$$

$$\rho(\vec{r},t) = \psi^*(\vec{r}_1,t_1)\psi(\vec{r}_2,t_2)$$

$$\begin{aligned}\vec{j}(\vec{r},t) &= \frac{-i\hbar}{2m}(\psi^*(\vec{r},t)\nabla\psi(\vec{r},t) - (\nabla\psi^*(\vec{r},t))\psi(\vec{r},t)) \\ &\quad - \frac{e\vec{A}}{mc}|\psi(\vec{r},t)|^2.\end{aligned}$$

$$H = \frac{(\vec{P} - e\vec{A}/c)^2}{2m} + e\Phi,$$

$$\text{For } V = \beta\delta(x-y): \quad -\frac{\hbar^2}{2m}\left(\frac{\partial}{\partial x}\psi(x)|_{y+\epsilon} - \frac{\partial}{\partial x}\psi(x)|_{y-\epsilon}\right) = -\beta\psi(y),$$

$$\vec{E} = -\nabla\Phi - \frac{1}{c}\partial_t\vec{A}, \quad \vec{B} = \nabla\times\vec{A},$$

$$\omega_{\text{cyclotron}} = \frac{eB}{mc},$$

$$e^{A+B} = e^A e^B e^{-C/2}, \quad \text{if } [A,B] = C, \text{ and } [C,A] = [C,B] = 0,$$

$$Y_{0,0} = \frac{1}{\sqrt{4\pi}}, \quad Y_{1,0} = \sqrt{\frac{3}{4\pi}}\cos\theta, \quad Y_{1,\pm1} = \mp\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\pm\phi},$$

$$Y_{2,0} = \sqrt{\frac{5}{16\pi}}(3\cos^2\theta - 1), \quad Y_{2,\pm1} = \mp\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta e^{\pm i\phi},$$

$$Y_{2,\pm2} = \sqrt{\frac{15}{32\pi}}\sin^2\theta e^{\pm 2i\phi}, \quad Y_{\ell-m}(\theta,\phi) = (-1)^m Y^*_{\ell m}(\theta,\phi).$$

$$|N\rangle = |n\rangle - \sum_{m\neq n} |m\rangle \frac{1}{\epsilon_m - \epsilon_n} \langle m|V|n\rangle + \cdots$$

$$E_N = \epsilon_n + \langle n|V|n\rangle - \sum_{m\neq n} \frac{|\langle m|V|n\rangle|^2}{\epsilon_m - \epsilon_n}$$

$$j_0(x) = \frac{\sin x}{x},\ n_0(x) = -\frac{\cos x}{x},\ j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x},\ n_1(x) = -\frac{\cos x}{x^2} - \frac{\sin x}{x}$$

$$j_2(x) = \left(\frac{3}{x^3} - \frac{1}{x}\right)\sin x - \frac{3}{x^2}\cos x,\ n_2(x) = -\left(\frac{3}{x^3} - \frac{1}{x}\right)\cos x - \frac{3}{x^2}\sin x,$$

$$\frac{d}{dt}P_{i\to n}(t) = \frac{2\pi}{\hbar}|V_{ni}|^2\delta(E_n - E_i),$$

$$\frac{d\sigma}{d\Omega} = \frac{m^2}{4\pi^2\hbar^4}\left|\int d^3r\mathcal{V}(r)e^{i(\vec{k}_f - \vec{k}_i)\cdot\vec{r}}\right|^2,$$

$$\sigma = \frac{(2S_R+1)}{(2S_1+1)(2S_2+1)}\frac{4\pi}{k^2}\frac{(\hbar\Gamma_R/2)^2}{(\epsilon_k - \epsilon_r)^2 + (\hbar\Gamma_R/2)^2},$$

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_{\rm single}\tilde{S}(\vec{q}),\quad \tilde{S}(\vec{q}) = \left|\sum_{\vec{a}} e^{i\vec{q}\cdot\vec{a}}\right|^2,$$

$$e^{i\vec{k}\cdot\vec{r}} = \sum_\ell (2\ell+1)i^\ell j_\ell(kr)P_\ell(\cos\theta),$$

$$P_\ell(\cos\theta) = \sqrt{\frac{4\pi}{2\ell+1}}Y_{\ell,m=0}(\theta,\phi),$$

$$P_0(x) = 1,\ P_1(x) = x,\ P_2(x) = (3x^2-1)/3,$$

$$f(\Omega) \equiv \sum_\ell (2\ell+1)e^{i\delta_\ell}\sin\delta_\ell\frac{1}{k}P_\ell(\cos\theta)$$

$$\psi_{\vec{k}}(\vec{r})|_{R\to\infty} = e^{i\vec{k}\cdot\vec{r}} + \frac{e^{ikr}}{r}f(\Omega),$$

$$\frac{d\sigma}{d\Omega} = |f(\Omega)|^2,\quad \sigma = \frac{4\pi}{k^2}\sum_\ell (2\ell+1)\sin^2\delta_\ell,\quad \delta \approx -ak$$

$$L_\pm|\ell,m\rangle = \sqrt{\ell(\ell+1) - m(m\pm1)}|\ell,m\pm1\rangle,$$

$$C^{\ell,s}_{m_\ell,m_s;JM} = \langle \ell,s,J,M|\ell,s,m_\ell,m_s\rangle,$$

$$\langle\tilde{\beta},J,M|T^k_q|\beta,\ell,m_\ell\rangle = C^{k\ell}_{qm_\ell;JM}\frac{\langle\tilde{\beta},J||T^{(k)}||\beta,\ell,J\rangle}{\sqrt{2J+1}},$$

$$n = \frac{(2s+1)}{(2\pi)^d}\int_{k<k_f} d^dk,\quad d\text{ dimensions},$$

$$\{\Psi_s(\vec{x}),\Psi^\dagger_{s'}(\vec{y})\} = \delta^3(\vec{x}-\vec{y})\delta_{ss'},$$

$$\Psi^\dagger_s(\vec{r}) = \frac{1}{\sqrt{V}}\sum_{\vec{k}} e^{i\vec{k}\cdot\vec{r}}a^\dagger_s(\vec{k}),\quad \{\Psi_s(\vec{x}),a^\dagger_\alpha\} = \phi_{\alpha,s}(\vec{x}).$$

1. An electron is in the $|\uparrow\rangle$ (up along the z axis) state.

   (a) (10 pts) What is the probability of finding it in the $|\uparrow\rangle$ state if observed after rotating the coordinate system by 90 degrees about the z axis?

   (b) (10 pts) If the rotation were instead 180 degrees about the x axis, what would the probability of observing the $|\uparrow\rangle$ state be?

a) 100 %

b) $|\psi\rangle = e^{i\sigma_x \varphi/2} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$= \cos\frac{\varphi}{2} + i\sin\frac{\varphi}{2}\,\sigma_x \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$= \begin{pmatrix} \cos\frac{\varphi}{2} \\ i\sin\frac{\varphi}{2} \end{pmatrix}$

$\text{Prob} = \cos^2\frac{\varphi}{2}$, $\varphi = 180^\circ$

$\text{Prob} = 0$

2. (20 pts) Consider a two-state system, with the Hamiltonian,

$$H = A + B\vec{\sigma}\cdot\hat{n},$$

where $\hat{n}$ is a unit vector in the $x - z$ plane pointing 30 degrees from the z axis. At $t = 0$, the state

$$|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

is occupied. Find the probability the state $|\downarrow\rangle$ is occupied as a function of time.

$$e^{-iHt} = e^{-iAt/\hbar} e^{-iB\vec{\sigma}\cdot\hat{n}\,t/\hbar}$$

$$= e^{-iAt/\hbar}\left\{\cos(Bt/\hbar) - i(\vec{\sigma}\cdot\hat{n})\sin(Bt/\hbar)\right\}$$

$$\vec{\sigma}\cdot\hat{n} = \sqrt{\tfrac{3}{2}}\,\sigma_z + \frac{1}{2}\sigma_x$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix}^{+} e^{-iHt}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = e^{-iAt/\hbar}\left(\frac{-i}{2}\sin Bt/\hbar\right)$$

$$\text{Prob} = \frac{1}{4}\sin^2 Bt/\hbar$$

3. (20 pts) A particle of mass $\boldsymbol{m}$ is in the first excited state of a harmonic oscillator with characteristic frequency $\boldsymbol{\omega}$. The well suddenly dissolves. What is the differential probability, $\boldsymbol{dN/dp}$, for observing the particle with momentum $\boldsymbol{p}$?

$$\mathbb{P}_p = \left| \langle p | \psi_1 \rangle \right|^2$$

$$= \left| \int dx\, e^{-ipx/\hbar} \psi_1(x) \right|^2$$

$$\psi_0(x) = \frac{1}{(\pi b^2)^{1/4}} e^{-x^2/2b^2}, \quad b^2 = \frac{\hbar}{m\omega},$$

$$\psi_1(x) = \left(\sqrt{\frac{m\omega}{2\hbar}}\, x - i \sqrt{\frac{1}{2m\hbar\omega}} (-i\hbar \partial_x) \right) \psi$$

$$= \sqrt{2}\,\frac{x}{b}\, e^{-x^2/2b^2} \Big/ (\pi b^2)^{1/4}$$

$$P_p = \frac{2}{\sqrt{\pi}\, b^3} \left| \frac{1}{\sqrt{L}} \int e^{-ipx/\hbar} dx\, x\, e^{-x^2/2b^2} \right|^2$$

$$= \frac{2}{\sqrt{\pi}\, b^3 L} \left| i\hbar \partial_p \int dx\, e^{-ipx/\hbar} e^{-x^2/2b^2} \right|^2$$

$$= \frac{2}{\sqrt{\pi}\, b^3 L} \left| i\hbar \partial_p \int dx\, e^{-\frac{(x + \frac{ip}{\hbar} b^2)^2}{2b^2}} e^{-\frac{p^2 b^2}{2\hbar^2}} \right|^2.$$

$$= \frac{2\hbar^2}{\sqrt{\pi}\, b^3 L} \left| \partial_p \left(\sqrt{2\pi b^2}\, e^{-p^2 b^2/2\hbar^2} \right) \right|^2$$

$$= \frac{2\hbar^2}{\sqrt{\pi}\, b^3 L} \left| \sqrt{2\pi}\, b\, \frac{p b^2}{\hbar^2} e^{-p^2 b^2/2\hbar^2} \right|^2 = \frac{4\sqrt{\pi}\, b^3}{L \hbar^2} p^2 e^{-\frac{p^2 b^2}{\hbar^2}}$$

$$P_p = \frac{4\sqrt{\pi}\, b^3}{L \hbar^2}\, p^2 e^{-\frac{p^2 b^2}{\hbar^2}} = \text{Prob to go into a single state}$$

$$\frac{dN}{dp} = P_p \cdot \frac{dN_{\text{state}}}{dp} = P_p \cdot \frac{L}{2\pi\hbar}$$

$$= \frac{2\, b^3 p^2}{\sqrt{\pi}\ \hbar^3}\, e^{-p^2 b^2/\hbar^2}$$

where $b = \sqrt{\hbar/m\omega}$

$$\int p^2\, dp\, e^{-p^2 b^2/\hbar^2}$$

$$= \left(\frac{\hbar}{\sqrt{2}\, b}\right)^3 (2\pi)^{1/2} = \frac{\sqrt{\pi}\, \hbar^3}{2 b^3}$$

4. (15 pts) Consider a particle of mass $\boldsymbol{m}$ confined to two-dimensions which feels the potential,

$$\boldsymbol{V(x,y) = \frac{1}{2}m\omega^2(x^2+y^2).}$$

What are the energy levels and their degeneracies?

$$E = (n_x + n_y + 1)\hbar\omega$$

$$= N\hbar\omega, \quad N = 1, 2, 3 \dots$$

$$d(N) = N$$

1 way for $N=1$

2 ways for $N=2$

3 " " $N=3$

⋮

5. Consider a particle of mass m approaching a positive potential step from the $x = +\infty$ direction with wave number k. The potential step is larger than the kinetic energy of the approaching particle, and has the form,

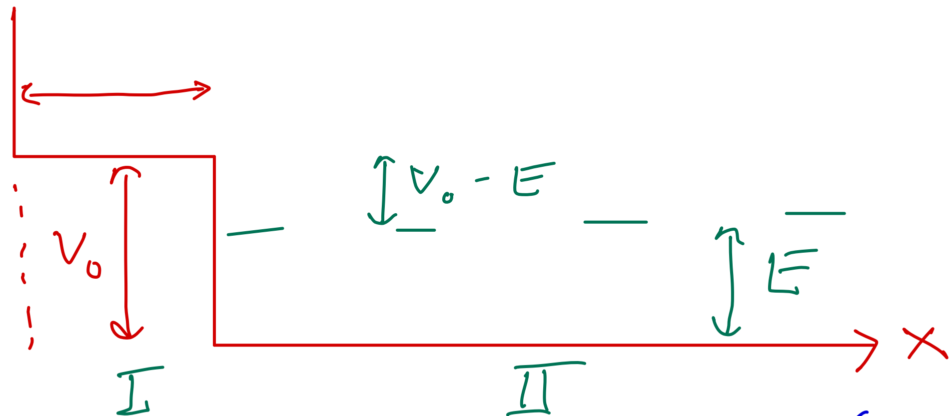
$$V(x) = \begin{cases} \infty, & x < 0 \\ V_0, & 0 < x < a \\ 0, & x > a \end{cases} .$$

For $x > a$, the wave function has the form

$$\psi(x) = e^{-ikx} - e^{2i\delta(k)} e^{ikx}.$$

(a) (20 pts) Find the phase shift $\delta(k)$.

(b) (5 pts) What is $\delta(k)$ in the limit $V_0 \to \infty$?



(a) $\psi_I = A \sinh qx, \quad \psi_{II} = \sin(kx + \delta)$

$$q = \sqrt{2m(V_0 - E)/\hbar^2}, \quad k = \sqrt{2mE/\hbar^2}$$

$$A \sinh qa = \sin(ka + \delta)$$

$$qA \cosh qa = k \cos(ka + \delta)$$

$$\frac{1}{q} \tanh qa = \frac{1}{k} \tan(ka + \delta)$$

$$\delta = -ka + \tan^{-1} \frac{k}{q} \tanh qa$$

(b) As $V_0 \to \infty$, $q \to \infty$

$$\delta = -ka \qquad (\psi_I = \sin(kx - a))$$