PHY 851 – QUANTUM MECHANICS
MIDTERM I
September 25-28, 2020

- Submit your exam via GRADESCOPE. Be sure that no page contains responses for more than one problem. Having different parts 2b,2c$\cdots$ of a given problem on a single page is fine. Individual problems can span multiple pages.
- Once the exam is opened, you are to upload your answer within six hours.
- This exam is open-book, open-notes, open-internet but closed-mouth. You are permitted to use mathematical software, e.g. Mathematica. You are not to communicate with any other individuals regarding the exam, during the exam.

$$\int_{-\infty}^{\infty} dx\ e^{-x^2/(2a^2)} = a\sqrt{2\pi},$$

$$H = i\hbar\partial_t,\ \vec{P} = -i\hbar\nabla,$$

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

$$U(t,-\infty) = 1 + \frac{-i}{\hbar}\int_{-\infty}^{t} dt'\ V(t')U(t',-\infty),$$

$$\langle x|x'\rangle = \delta(x-x'),\ \langle p|p'\rangle = \frac{1}{2\pi\hbar}\delta(p-p'),$$

$$|p\rangle = \int dx\ |x\rangle e^{ipx/\hbar},\quad |x\rangle = \int \frac{dp}{2\pi\hbar}|p\rangle e^{-ipx/\hbar},$$

$$H = \frac{P^2}{2m} + \frac{1}{2}m\omega^2x^2 = \hbar\omega(a^\dagger a + 1/2),$$

$$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}X - i\sqrt{\frac{1}{2\hbar m\omega}}P,$$

$$\psi_0(x) = \frac{1}{(\pi b^2)^{1/4}}e^{-x^2/2b^2},\quad b^2 = \frac{\hbar}{m\omega},$$

$$\rho(\vec{r},t) = \psi^*(\vec{r}_1,t_1)\psi(\vec{r}_2,t_2)$$

$$\begin{aligned}\vec{j}(\vec{r},t) &= \frac{-i\hbar}{2m}(\psi^*(\vec{r},t)\nabla\psi(\vec{r},t) - (\nabla\psi^*(\vec{r},t))\psi(\vec{r},t)) \\ &- \frac{e\vec{A}}{mc}|\psi(\vec{r},t)|^2.\end{aligned}$$

$$H = \frac{(\vec{P}-e\vec{A}/c)^2}{2m} + e\Phi,$$

$$\text{For } V = \beta\delta(x-y): \quad -\frac{\hbar^2}{2m}\left(\frac{\partial}{\partial x}\psi(x)|_{y+\epsilon} - \frac{\partial}{\partial x}\psi(x)|_{y-\epsilon}\right) = -\beta\psi(y),$$

$$\vec{E} = -\nabla\Phi - \frac{1}{c}\partial_t\vec{A},\quad \vec{B} = \nabla\times\vec{A},$$

$$\omega_{\text{cyclotron}} = \frac{eB}{mc},$$

$$e^{A+B} = e^Ae^Be^{-C/2},\ \text{ if } [A,B] = C,\ \text{ and } [C,A] = [C,B] = 0,$$

$$Y_{0,0} = \frac{1}{\sqrt{4\pi}},\quad Y_{1,0} = \sqrt{\frac{3}{4\pi}}\cos\theta,\quad Y_{1,\pm1} = \mp\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\pm\phi},$$

$$Y_{2,0} = \sqrt{\frac{5}{16\pi}}(3\cos^2\theta - 1),\quad Y_{2,\pm1} = \mp\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta e^{\pm i\phi},$$

$$Y_{2,\pm2} = \sqrt{\frac{15}{32\pi}}\sin^2\theta e^{\pm2i\phi},\quad Y_{\ell-m}(\theta,\phi) = (-1)^mY^*_{\ell m}(\theta,\phi).$$

$$|N\rangle = |n\rangle - \sum_{m\neq n} |m\rangle \frac{1}{\epsilon_m - \epsilon_n} \langle m|V|n\rangle + \cdots$$

$$E_N = \epsilon_n + \langle n|V|n\rangle - \sum_{m\neq n} \frac{|\langle m|V|n\rangle|^2}{\epsilon_m - \epsilon_n}$$

$$j_0(x) = \frac{\sin x}{x},\ n_0(x) = -\frac{\cos x}{x},\ j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x},\ n_1(x) = -\frac{\cos x}{x^2} - \frac{\sin x}{x}$$

$$j_2(x) = \left(\frac{3}{x^3} - \frac{1}{x}\right)\sin x - \frac{3}{x^2}\cos x,\ n_2(x) = -\left(\frac{3}{x^3} - \frac{1}{x}\right)\cos x - \frac{3}{x^2}\sin x,$$

$$\frac{d}{dt} P_{i\to n}(t) = \frac{2\pi}{\hbar} |V_{ni}|^2 \delta(E_n - E_i),$$

$$\frac{d\sigma}{d\Omega} = \frac{m^2}{4\pi^2\hbar^4} \left| \int d^3r \mathcal{V}(r) e^{i(\vec{k}_f - \vec{k}_i)\cdot\vec{r}} \right|^2,$$

$$\sigma = \frac{(2S_R+1)}{(2S_1+1)(2S_2+1)} \frac{4\pi}{k^2} \frac{(\hbar\Gamma_R/2)^2}{(\epsilon_k - \epsilon_r)^2 + (\hbar\Gamma_R/2)^2},$$

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_{\rm single} \tilde{S}(\vec{q}), \quad \tilde{S}(\vec{q}) = \left|\sum_{\vec{a}} e^{i\vec{q}\cdot\vec{a}}\right|^2,$$

$$e^{i\vec{k}\cdot\vec{r}} = \sum_\ell (2\ell+1) i^\ell j_\ell(kr) P_\ell(\cos\theta),$$

$$P_\ell(\cos\theta) = \sqrt{\frac{4\pi}{2\ell+1}} Y_{\ell,m=0}(\theta,\phi),$$

$$P_0(x) = 1,\ P_1(x) = x,\ P_2(x) = (3x^2-1)/3,$$

$$f(\Omega) \equiv \sum_\ell (2\ell+1) e^{i\delta_\ell} \sin\delta_\ell \frac{1}{k} P_\ell(\cos\theta)$$

$$\psi_{\vec{k}}(\vec{r})|_{R\to\infty} = e^{i\vec{k}\cdot\vec{r}} + \frac{e^{ikr}}{r} f(\Omega),$$

$$\frac{d\sigma}{d\Omega} = |f(\Omega)|^2, \quad \sigma = \frac{4\pi}{k^2} \sum_\ell (2\ell+1)\sin^2\delta_\ell, \quad \delta \approx -ak$$

$$L_\pm|\ell,m\rangle = \sqrt{\ell(\ell+1) - m(m\pm1)}|\ell, m\pm1\rangle,$$

$$C^{\ell,s}_{m_\ell,m_s;JM} = \langle \ell,s,J,M|\ell,s,m_\ell,m_s\rangle,$$

$$\langle\tilde{\beta},J,M|T^k_q|\beta,\ell,m_\ell\rangle = C^{k\ell}_{qm_\ell;JM} \frac{\langle\tilde{\beta},J||T^{(k)}||\beta,\ell,J\rangle}{\sqrt{2J+1}},$$

$$n = \frac{(2s+1)}{(2\pi)^d} \int_{k<k_f} d^dk, \quad d \text{ dimensions},$$

$$\{\Psi_s(\vec{x}), \Psi^\dagger_{s'}(\vec{y})\} = \delta^3(\vec{x}-\vec{y})\delta_{ss'},$$

$$\Psi^\dagger_s(\vec{r}) = \frac{1}{\sqrt{V}} \sum_{\vec{k}} e^{i\vec{k}\cdot\vec{r}} a^\dagger_s(\vec{k}), \quad \{\Psi_s(\vec{x}), a^\dagger_\alpha\} = \phi_{\alpha,s}(\vec{x}).$$

1. An electron is in the $|\uparrow\rangle$ (up along the $\boldsymbol{z}$ axis) state.

   (a) (10 pts) What is the probability of finding it in the $|\uparrow\rangle$ state if observed after rotating the coordinate system by 90 degrees about the $\boldsymbol{z}$ axis?

   (b) (10 pts) If the rotation were instead 180 degrees about the $\boldsymbol{x}$ axis, what would the probability of observing the $|\uparrow\rangle$ state be?

2. (20 pts) Consider a two-state system, with the Hamiltonian,

$$\boldsymbol{H = A + B\vec{\sigma} \cdot \hat{n},}$$

where $\boldsymbol{\hat{n}}$ is a unit vector in the $\boldsymbol{x - z}$ plane pointing 30 degrees from the $\boldsymbol{z}$ axis. At $\boldsymbol{t = 0}$, the state

$$|\uparrow\rangle = \begin{pmatrix} \mathbf{1} \\ \mathbf{0} \end{pmatrix}$$

is occupied. Find the probability the state $|\downarrow\rangle$ is occupied as a function of time.

3. (20 pts) A particle of mass $\boldsymbol{m}$ is in the first excited state of a harmonic oscillator with characteristic frequency $\boldsymbol{\omega}$. The well suddenly dissolves. What is the differential probability, $\boldsymbol{dN/dp}$, for observing the particle with momentum $\boldsymbol{p}$?

4. (15 pts) Consider a particle of mass m confined to two-dimensions which feels the potential,

$$V(x,y) = \frac{1}{2} m\omega^2 (x^2 + y^2).$$

What are the energy levels and their degeneracies?

5. Consider a particle of mass m approaching a positive potential step from the $x = +\infty$ direction with wave number k. The potential step is larger than the kinetic energy of the approaching particle, and has the form,

$$V(x) = \begin{cases} \infty, & x < 0 \\ V_0, & 0 < x < a \\ 0, & x > a \end{cases}.$$

For $x > a$, the wave function has the form

$$\psi(x) = e^{-ikx} - e^{2i\delta(k)} e^{ikx}.$$

(a) (20 pts) Find the phase shift $\delta(k)$.

(b) (5 pts) What is $\delta(k)$ in the limit $V_0 \to \infty$?