*FINAL (SUBJECT) EXAM,*
**PHYSICS 852, Spring 2020**
May 29-30, 3:00-3:00 PM

SECRET STUDENT NUMBER:
STUDNUMBER

1. Solutions should be submitted through Gradescope within 20 minutes of the finishing time (6PM).
2. **DO NOT WRITE YOUR NAME ANYWHERE ON THE EXAM. Your exam has a "secret student number" on each page. Write that number on EACH page of the solutions you submit**.
3. This exam is closed-book and closed-note. You are not permitted to use mathematical software, e.g. Mathematica.
4. This exam has five problems with a summed valued of 100 points.

$$\int_{-\infty}^{\infty} dx\; e^{-x^2/(2a^2)} = a\sqrt{2\pi},$$

$$H = i\hbar\partial_t,\;\; \vec{P} = -i\hbar\nabla,$$

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

$$U(t,-\infty) = 1 + \frac{-i}{\hbar}\int_{-\infty}^{t} dt'\; V(t')U(t',-\infty),$$

$$\langle x|x'\rangle = \delta(x-x'),\; \langle p|p'\rangle = \frac{1}{2\pi\hbar}\delta(p-p'),$$

$$|p\rangle = \int dx\; |x\rangle e^{ipx/\hbar}, \quad |x\rangle = \int \frac{dp}{2\pi\hbar}|p\rangle e^{-ipx/\hbar},$$

$$H = \frac{P^2}{2m} + \frac{1}{2}m\omega^2 x^2 = \hbar\omega(a^\dagger a + 1/2),$$

$$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}X - i\sqrt{\frac{1}{2\hbar m\omega}}P,$$

$$\psi_0(x) = \frac{1}{(\pi b^2)^{1/4}}e^{-x^2/2b^2}, \quad b^2 = \frac{\hbar}{m\omega},$$

$$\rho(\vec{r},t) = \psi^*(\vec{r}_1,t_1)\psi(\vec{r}_2,t_2)$$

$$\vec{j}(\vec{r},t) = \frac{-i\hbar}{2m}(\psi^*(\vec{r},t)\nabla\psi(\vec{r},t) - (\nabla\psi^*(\vec{r},t))\psi(\vec{r},t)) - \frac{e\vec{A}}{mc}|\psi(\vec{r},t)|^2.$$

$$H = \frac{(\vec{P} - e\vec{A}/c)^2}{2m} + e\Phi,$$

For $V = \beta\delta(x-y)$ : $\quad -\frac{\hbar^2}{2m}\left(\frac{\partial}{\partial x}\psi(x)|_{y+\epsilon} - \frac{\partial}{\partial x}\psi(x)|_{y-\epsilon}\right) = -\beta\psi(y),$

$$\vec{E} = -\nabla\Phi - \frac{1}{c}\partial_t\vec{A}, \quad \vec{B} = \nabla\times\vec{A},$$

$$\omega_{\text{cyclotron}} = \frac{eB}{mc},$$

$$e^{A+B} = e^A e^B e^{-C/2}, \;\text{ if } [A,B] = C, \text{ and } [C,A] = [C,B] = 0,$$

$$Y_{0,0} = \frac{1}{\sqrt{4\pi}}, \quad Y_{1,0} = \sqrt{\frac{3}{4\pi}}\cos\theta, \quad Y_{1,\pm1} = \mp\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\pm\phi},$$

$$Y_{2,0} = \sqrt{\frac{5}{16\pi}}(3\cos^2\theta - 1), \quad Y_{2,\pm1} = \mp\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta e^{\pm i\phi},$$

$$Y_{2,\pm2} = \sqrt{\frac{15}{32\pi}}\sin^2\theta e^{\pm 2i\phi}, \quad Y_{\ell-m}(\theta,\phi) = (-1)^m Y^*_{\ell m}(\theta,\phi).$$

$$x = \frac{1}{\sqrt{2}}\left(Y_{1-1} - Y_{11}\right) < \begin{array}{l} Y_{11} = -\frac{1}{\sqrt{2}}(x+iy) \\ Y_{1-1} = \frac{1}{\sqrt{2}}(x-iy) \end{array}$$

$$|N\rangle = |n\rangle - \sum_{m\neq n} |m\rangle \frac{1}{\epsilon_m - \epsilon_n} \langle m|V|n\rangle + \cdots$$

$$E_N = \epsilon_n + \langle n|V|n\rangle - \sum_{m\neq n} \frac{|\langle m|V|n\rangle|^2}{\epsilon_m - \epsilon_n}$$

$$j_0(x) = \frac{\sin x}{x},\ n_0(x) = -\frac{\cos x}{x},\ j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x},\ n_1(x) = -\frac{\cos x}{x^2} - \frac{\sin x}{x}$$

$$j_2(x) = \left(\frac{3}{x^3} - \frac{1}{x}\right)\sin x - \frac{3}{x^2}\cos x,\ n_2(x) = -\left(\frac{3}{x^3} - \frac{1}{x}\right)\cos x - \frac{3}{x^2}\sin x,$$

$$\frac{d}{dt} P_{i\to n}(t) = \frac{2\pi}{\hbar} |V_{ni}|^2 \delta(E_n - E_i),$$

$$\frac{d\sigma}{d\Omega} = \frac{m^2}{4\pi^2\hbar^4} \left| \int d^3 r \mathcal{V}(r) e^{i(\vec{k}_f - \vec{k}_i)\cdot\vec{r}} \right|^2,$$

$$\sigma = \frac{(2S_R+1)}{(2S_1+1)(2S_2+1)} \frac{4\pi}{k^2} \frac{(\hbar\Gamma_R/2)^2}{(\epsilon_k - \epsilon_r)^2 + (\hbar\Gamma_R/2)^2},$$

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_{\text{single}} \tilde{S}(\vec{q}),\quad \tilde{S}(\vec{q}) = \left| \sum_{\vec{a}} e^{i\vec{q}\cdot\vec{a}} \right|^2,$$

$$e^{i\vec{k}\cdot\vec{r}} = \sum_\ell (2\ell+1) i^\ell j_\ell(kr) P_\ell(\cos\theta),$$

$$P_\ell(\cos\theta) = \sqrt{\frac{4\pi}{2\ell+1}} Y_{\ell,m=0}(\theta,\phi),$$

$$P_0(x) = 1,\ P_1(x) = x,\ P_2(x) = (3x^2-1)/3,$$

$$f(\Omega) \equiv \sum_\ell (2\ell+1) e^{i\delta_\ell} \sin\delta_\ell \frac{1}{k} P_\ell(\cos\theta)$$

$$\psi_{\vec{k}}(\vec{r})|_{R\to\infty} = e^{i\vec{k}\cdot\vec{r}} + \frac{e^{ikr}}{r} f(\Omega),$$

$$\frac{d\sigma}{d\Omega} = |f(\Omega)|^2,\quad \sigma = \frac{4\pi}{k^2} \sum_\ell (2\ell+1)\sin^2\delta_\ell,\quad \delta \approx -ak$$

$$L_\pm |\ell, m\rangle = \sqrt{\ell(\ell+1) - m(m\pm 1)} |\ell, m\pm 1\rangle,$$

$$C^{\ell,s}_{m_\ell,m_s;JM} = \langle \ell, s, J, M | \ell, s, m_\ell, m_s\rangle,$$

$$\langle \tilde{\beta}, J, M | T^k_q | \beta, \ell, m_\ell\rangle = C^{k\ell}_{qm_\ell;JM} \frac{\langle \tilde{\beta}, J || T^{(k)} || \beta, \ell, J\rangle}{\sqrt{2J+1}},$$

$$n = \frac{(2s+1)}{(2\pi)^d} \int_{k<k_f} d^d k,\quad d \text{ dimensions},$$

$$\{\Psi_s(\vec{x}), \Psi^\dagger_{s'}(\vec{y})\} = \delta^3(\vec{x} - \vec{y})\delta_{ss'},$$

$$\Psi^\dagger_s(\vec{r}) = \frac{1}{\sqrt{V}} \sum_{\vec{k}} e^{i\vec{k}\cdot\vec{r}} a^\dagger_s(\vec{k}),\quad \{\Psi_s(\vec{x}), a^\dagger_\alpha\} = \phi_{\alpha,s}(\vec{x}).$$

1. A particle of mass $m$ in the first excited state of a one-dimensional harmonic oscillator decays to the ground state. The harmonic oscillator wave functions are:

$$\psi_0(x) = \frac{1}{\pi^{1/4}a^{1/2}}e^{-x^2/2a^2},$$
$$\psi_1(x) = 2^{1/2}(x/a)\psi_0(x),$$

where $a^2 = \hbar/(m\omega)$ and the binding energy is $\hbar\omega$. The decay proceeds through the interaction

$$V = g\int dx\Phi(x)\Psi^\dagger(x)\Psi(x),$$

where $g$ is a coupling constant and $\Psi^\dagger$ and $\Psi$ are field operators for the particle in the harmonic oscillator defined such that

$$\left[\Psi(x), \Psi^\dagger(x')\right] = \delta(x - x').$$

The field operator $\Phi$ creates or destroys massless particles and is defined as

$$\Phi = \sum_k \sqrt{\frac{\hbar}{2E_k L}}\left[a_k e^{ikx} + a_k^\dagger e^{-ikx}\right],$$

where $L$ is an arbitrarily large length and $E_k = \hbar ck$.

(a) (10 pts) Write down an integral $I$ such that the matrix element for the transition to the ground state,

$$\mathcal{M} \equiv \langle 0, k|V|1\rangle = \frac{1}{\sqrt{L}}I.$$

DO NOT evaluate the integral!!!

(b) (10 pts) Using Fermi's golden rule, find the decay rate in terms of $I, m$ and $\omega$.

a) $\mathcal{M} = \langle 0, k | V | 1 \rangle$

$$= g \int dx\, \psi_1(x) \psi_0(x) \cdot \langle k | \Phi(x) | 0 \rangle$$

$$= g \sqrt{\frac{\hbar}{2E_k L}} \int dx\, \psi_1(x) \psi_0(x)\, e^{-ikx}$$

$$I(k) = g \sqrt{\frac{\hbar}{2E_k}} \int dx\, e^{-ikx} \psi_1(x) \psi_0(x)$$

b) $\Gamma = \frac{2\pi}{\hbar} \sum_k |\mathcal{M}|^2 \delta(\varepsilon_k - \hbar\omega)$

$$= \frac{2\pi}{\hbar} \frac{L}{\pi} \int_0^\infty dk\, \frac{1}{L} I^2 \delta(\varepsilon_k - \hbar\omega)$$

$$= \frac{2}{\hbar} I^2(k) \frac{1}{d\varepsilon_k / dk} = \frac{2}{\hbar} I^2(k) \frac{1}{\hbar c}$$

$$= \frac{2}{\hbar^2 c} I^2$$

2. A neutron and a proton populate the ground state of a harmonic oscillator. The proton and neutron interact via a spin-spin interaction,

$$V_{\text{s.s.}} = -\alpha \vec{S}_p \cdot \vec{S}_n.$$

Additionally, a magnetic interaction is added,

$$V_B = -\beta_n \vec{S}_n \cdot \vec{B} - \beta_p \vec{S}_p \cdot \vec{B}.$$

Work in the following basis:

$$|S=1, M_S=1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, |S=1, M_S=-1\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},$$
$$|S=1, M_S=0\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, |S=0, M_S=0\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

(a) (15 pts) Write the Hamiltonian, $V_{\text{s.s.}} + V_B$, as a $4 \times 4$ matrix in this basis.

(b) (5 pts) What are the eigen-energies?

$$V_{ss} = -\frac{\alpha}{2}\left((\vec{S}_p+\vec{S}_n)^2 - \vec{S}_p^{\,2} - \vec{S}_n^{\,2}\right)\hbar^2$$

$$= -\frac{\alpha}{2}\left\{ S(S+1) - \frac{3}{2}\right\}\hbar^2$$

$$= \begin{pmatrix} -\alpha/4 & 0 & 0 & 0 \\ 0 & -\alpha/4 & 0 & 0 \\ 0 & 0 & -\alpha/4 & 0 \\ 0 & 0 & 0 & \frac{3\alpha}{4} \end{pmatrix}\hbar^2$$

$$\langle S=1, M=1 | V_B | S=1, M=1\rangle =$$

$$\langle m_n = \tfrac{1}{2}, m_p = \tfrac{1}{2} | V_B | m_n = \tfrac{1}{2}, m_n = \tfrac{1}{2}\rangle$$

$$= -\frac{\hbar B}{2}(\beta_n + \beta_p)$$

$$\langle S=1, M=-1 | V_B | S=1, M=-1\rangle =$$

$$\langle m_n = -\tfrac{1}{2}, m_p = -\tfrac{1}{2} | V_B | m_n = -\tfrac{1}{2}, m_p = -\tfrac{1}{2}\rangle$$

$$= \frac{\hbar B}{2}(\beta_n + \beta_p)$$

$$\langle S=1, M=0 | V_B | S=1, M=0\rangle$$

$$= \frac{1}{\sqrt{2}}\left[\langle m_n = \tfrac{1}{2}, m_p = -\tfrac{1}{2}| + \langle m_n = -\tfrac{1}{2}, m_p = \tfrac{1}{2}|\right] V_B$$

$$\frac{1}{\sqrt{2}}\left[|m_n = \tfrac{1}{2}, m_p = -\tfrac{1}{2}\rangle + |m_n = -\tfrac{1}{2}, m_p = \tfrac{1}{2}\rangle\right]$$

$$= \frac{1}{2}\frac{\hbar B}{2}(-\beta_n + \beta_p) - \frac{1}{2}\frac{\hbar B}{2}(\beta_n - \beta_p) = 0$$

$$\langle S=0, M=0 | V_B | S=0, M=0\rangle$$

$$= \frac{1}{\sqrt{2}}\left[\langle m_n = \tfrac{1}{2}, m_p = -\tfrac{1}{2}| + \langle m_n = -\tfrac{1}{2}, m_p = \tfrac{1}{2}|\right] V_B$$

$$\frac{1}{\sqrt{2}}\left[|m_n = \tfrac{1}{2}, m_p = -\tfrac{1}{2}\rangle + |m_n = -\tfrac{1}{2}, m_p = \tfrac{1}{2}\rangle\right.$$

$$= \frac{1}{2}\frac{\hbar B}{2}(-\beta_n + \beta_p) - \frac{1}{2}\frac{\hbar B}{2}(\beta_n - \beta_p) = 0$$

$$\langle S=1, M=0 | V_B | S=0, M=0 \rangle$$
$$= \frac{1}{\sqrt{2}}\left[\langle m_n=\tfrac{1}{2}, m_p=\tfrac{1}{2}| + \langle m_n=-\tfrac{1}{2}, m_p=\tfrac{1}{2}|\right] V_B$$
$$\frac{1}{\sqrt{2}}\left[|m_n=\tfrac{1}{2}, m_p=-\tfrac{1}{2}\rangle - |m_n=-\tfrac{1}{2}, m_p=\tfrac{1}{2}\rangle\right]$$
$$= \frac{\hbar B}{4}\left\{-\beta_n+\beta_p - (\beta_n-\beta_p)\right\} = -\frac{\hbar B}{2}(\beta_n - \beta_p)$$
$$\langle S=0, M=0 | V_B | S=1, M=1 \rangle = \langle S=1, M=0 | V_B | S=0, M=0 \rangle^*$$
$$= -\hbar B(\beta_n - \beta_p)$$

$$H = \begin{pmatrix} -\frac{\alpha}{4} - \frac{\hbar B}{2}(\beta_n+\beta_p), & 0, & 0, & 0 \\ 0, & -\frac{\alpha}{4} + \frac{\hbar B}{2}(\beta_n+\beta_p), & 0, & 0 \\ 0, & 0, & -\alpha\hbar^2/4, & -\frac{\hbar B(\beta_n-\beta_p)}{2} \\ 0, & 0, & -\frac{\hbar B(\beta_n-\beta_p)}{2}, & \frac{3\alpha\hbar^2}{4} \end{pmatrix}$$

b)

$$E = \begin{cases} -\frac{\alpha\hbar^2}{4} - \frac{\hbar B}{2}(\beta_n+\beta_p) \\ -\frac{\alpha\hbar^2}{4} + \frac{\hbar B}{2}(\beta_n+\beta_p) \\ \alpha\hbar^2/4 + \sqrt{\frac{\alpha^2\hbar^4}{4} + \frac{\hbar^2 B^2(\beta_n-\beta_p)^2}{4}} \\ \frac{\alpha\hbar^2}{4} - \sqrt{\frac{\alpha^2\hbar^4}{4} + \frac{\hbar^2 B^2(\beta_n-\beta_p)^2}{4}} \end{cases}$$

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3. An external electric field, $\boldsymbol{E_0 \hat{x}}$, interacts with a system via the perturbative interaction

$$\boldsymbol{V = -q\vec{E}\cdot\vec{r}.}$$

$z = T^1_0$

$x = \frac{1}{\sqrt{2}}\left(-T^1_1 + T^1_{-1}\right)$

(a) (10 pts) ~~(5 pts each)~~ Circle the matrix elements which might be non-zero.

- $\boldsymbol{\langle \alpha, J=2, M_J=1|V|\beta, J=2, M_J=0\rangle}$
- $\boldsymbol{\langle \alpha, J=2, M_J=1|V|\beta, J=1, M_J=0\rangle}$
- $\boldsymbol{\langle \alpha, J=2, M_J=1|V|\beta, J=0, M_J=0\rangle}$.

(b) (10 pts) Bob and Carol perform a complicated integral to find the matrix element

$$\boldsymbol{I = \langle \alpha, J=1, M=0|z|\beta, J=0, M=0\rangle.}$$

Ted and Alice wish to calculate

$$\boldsymbol{J = \langle \alpha, J=1, M=1|x|\beta, J=0, M=0\rangle.}$$

Express $\boldsymbol{J}$ in terms of $\boldsymbol{I}$ and Clebsch-Gordan coefficients. You need not evaluate the Clebsch-Gordan coefficients.

$$z = T^1_0 \quad , \quad x = \frac{1}{\sqrt{2}}\left(T^1_{-1} - T^1_1\right)$$

$$I = \langle \alpha | J=1, M=0 | T^1_0 | \beta, J=0, M=0\rangle$$

$$J = -\langle \alpha | J=1, M=1 | \frac{1}{\sqrt{2}} T^1_1 | \beta, J=0, M=0\rangle$$

$$J = -\frac{1}{\sqrt{2}} \cdot I \cdot \frac{\langle J=1, M=1 | l_1=1, l_2=0, m_1=1, m_2=0\rangle}{\langle J=1, M=0 | l_1=1, l_2=0, m_1=0, m_2=0\rangle}$$

4. (~~15~~ 20 pts) Two identical scattering centers are located at $\boldsymbol{a}\hat{\boldsymbol{x}}$ and $-\boldsymbol{a}\hat{\boldsymbol{x}}$. A beam ~~traveling~~ of wavenumber $\boldsymbol{k}$ is traveling in the $\hat{\boldsymbol{z}}$ and interacts weakly with the scatterers. In terms of the polar angle $\boldsymbol{\theta}$ and the azimuthal angle $\boldsymbol{\phi}$, describe the directions for which the scattering amplitude vanishes.

$$S = e^{i\vec{q}\cdot\hat{x}\cdot a} + e^{-i\vec{q}\cdot\hat{x}\,a}$$

$$\vec{q} = k(1-\cos\theta)\hat{z} - \sin\theta\cos\phi\,\hat{x} - \sin\theta\sin\phi\,\hat{y}$$

$$S = e^{ika\sin\theta\cos\phi} + e^{-ika\sin\theta\cos\phi}$$

will vanish when

$$2ka\sin\theta\cos\phi = (2n+1)\pi$$

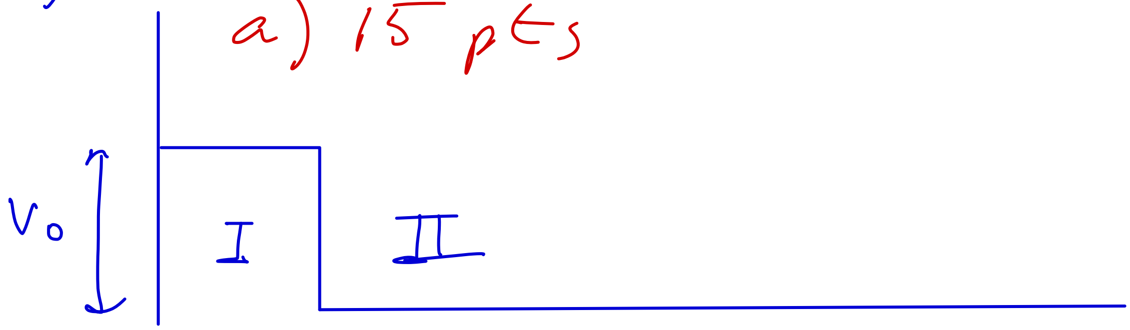
5. (15 pts) A particle of mass $m$ is incident on a spherically symmetric repulsive potential

$$V(r) = \begin{cases} V_0, & r < a \\ 0, & r > a \end{cases}.$$

a) Find $\delta(k)$ when incident energy is below barrier

b) As the incident energy approaches zero, what is the cross section?

a) 15 pts




$$\psi_I = A \sinh qr, \quad q = \sqrt{\frac{2m}{\hbar^2}(V_0 - E)}$$

$$\psi_{II} = \sin(kr + \delta)$$

$$A \sinh qa = \sin(ka + \delta)$$

$$qA \cosh qa = k \cos(ka + \delta)$$

$$\frac{1}{q} \tanh qa = \frac{1}{k} \tan(ka + \delta)$$

$$\delta = -ka + \tan^{-1}\left\{\frac{k}{q} \tanh qa\right\}$$

As $k \to 0$

$$\delta = -k\left\{a - \frac{\tanh qa}{q}\right\}$$

b) 5 pts

$$\sigma = \frac{4\pi}{k^2}\sin^2\delta = 4\pi\left[a - \frac{\tanh qa}{q}\right]^2$$

$$q = \sqrt{\frac{2m}{\hbar^2} V_0}$$

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