PHYSICS 851, FALL 2021

Tuesday, December 14, 12:45 PM

This exam is worth 100 points.

Your Name: ______________________________

$$\int_{-\infty}^{\infty} dx\; e^{-x^2/(2a^2)} = a\sqrt{2\pi},$$

$$H = i\hbar\partial_t,\;\; \vec{P} = -i\hbar\nabla,$$

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

$$U(t,-\infty) = 1 + \frac{-i}{\hbar}\int_{-\infty}^{t} dt'\; V(t')U(t',-\infty),$$

$$\langle x|x'\rangle = \delta(x-x'),\; \langle p|p'\rangle = \frac{1}{2\pi\hbar}\delta(p-p'),$$

$$|p\rangle = \int dx\; |x\rangle e^{ipx/\hbar},\quad |x\rangle = \int \frac{dp}{2\pi\hbar}|p\rangle e^{-ipx/\hbar},$$

$$H = \frac{P^2}{2m} + \frac{1}{2}m\omega^2 x^2 = \hbar\omega(a^\dagger a + 1/2),$$

$$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}X - i\sqrt{\frac{1}{2\hbar m\omega}}P,$$

$$\psi_0(x) = \frac{1}{(\pi b^2)^{1/4}}e^{-x^2/2b^2},\quad b^2 = \frac{\hbar}{m\omega},$$

$$\rho(\vec{r},t) = \psi^*(\vec{r}_1,t_1)\psi(\vec{r}_2,t_2)$$

$$\vec{j}(\vec{r},t) = \frac{-i\hbar}{2m}(\psi^*(\vec{r},t)\nabla\psi(\vec{r},t) - (\nabla\psi^*(\vec{r},t))\psi(\vec{r},t)) - \frac{e\vec{A}}{mc}|\psi(\vec{r},t)|^2.$$

$$H = \frac{(\vec{P} - e\vec{A}/c)^2}{2m} + e\Phi,$$

$$\text{For } V = \beta\delta(x-y):\quad -\frac{\hbar^2}{2m}\left(\partial_x\psi(x)|_{y+\epsilon} - \partial_x\psi(x)|_{y-\epsilon}\right) = -\beta\psi(y),$$

$$\vec{E} = -\nabla\Phi - \frac{1}{c}\partial_t\vec{A},\quad \vec{B} = \nabla\times\vec{A},$$

$$\omega_{\rm cyclotron} = \frac{eB}{mc},$$

$$e^{A+B} = e^A e^B e^{-C/2},\;\text{ if } [A,B] = C,\text{ and } [C,A] = [C,B] = 0,$$

$$Y_{0,0} = \frac{1}{\sqrt{4\pi}},\quad Y_{1,0} = \sqrt{\frac{3}{4\pi}}\cos\theta,\quad Y_{1,\pm1} = \mp\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\pm\phi},$$

$$Y_{2,0} = \sqrt{\frac{5}{16\pi}}(3\cos^2\theta - 1),\quad Y_{2,\pm1} = \mp\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta e^{\pm i\phi},$$

$$Y_{2,\pm2} = \sqrt{\frac{15}{32\pi}}\sin^2\theta e^{\pm 2i\phi},\quad Y_{\ell -m}(\theta,\phi) = (-1)^m Y^*_{\ell m}(\theta,\phi).$$

$$
\begin{aligned}
|N\rangle &= |n\rangle - \sum_{m\neq n} |m\rangle \frac{1}{\epsilon_m - \epsilon_n} \langle m|V|n\rangle + \cdots \\
E_N &= \epsilon_n + \langle n|V|n\rangle - \sum_{m\neq n} \frac{|\langle m|V|n\rangle|^2}{\epsilon_m - \epsilon_n}
\end{aligned}
$$

$$
j_0(x) = \frac{\sin x}{x},\ n_0(x) = -\frac{\cos x}{x},\ j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x},\ n_1(x) = -\frac{\cos x}{x^2} - \frac{\sin x}{x}
$$

$$
j_2(x) = \left(\frac{3}{x^3} - \frac{1}{x}\right)\sin x - \frac{3}{x^2}\cos x,\ n_2(x) = -\left(\frac{3}{x^3} - \frac{1}{x}\right)\cos x - \frac{3}{x^2}\sin x,
$$

$$
\begin{aligned}
\frac{d}{dt}P_{i\to n}(t) &= \frac{2\pi}{\hbar}|V_{ni}|^2\delta(E_n - E_i), \\
\frac{d\sigma}{d\Omega} &= \frac{m^2}{4\pi^2\hbar^4}\left|\int d^3r \mathcal{V}(r) e^{i(\vec{k}_f - \vec{k}_i)\cdot\vec{r}}\right|^2, \\
\sigma &= \frac{(2S_R + 1)}{(2S_1 + 1)(2S_2 + 1)} \frac{4\pi}{k^2} \frac{(\hbar\Gamma_R/2)^2}{(\epsilon_k - \epsilon_r)^2 + (\hbar\Gamma_R/2)^2}, \\
\frac{d\sigma}{d\Omega} &= \left(\frac{d\sigma}{d\Omega}\right)_{\text{single}} \tilde{S}(\vec{q}),\quad \tilde{S}(\vec{q}) = \frac{1}{N}\left|\sum_{\vec{a}} e^{i\vec{q}\cdot\vec{a}}\right|^2 = \sum_{\delta\vec{a}} e^{i\vec{q}\cdot\delta\vec{a}}, \\
\frac{d\sigma}{d\Omega} &= \left(\frac{d\sigma}{d\Omega}\right)_{\text{Rutherford}} \left|\frac{1}{e}\int d^3r\ \rho(\vec{r}) e^{i\vec{q}\cdot\vec{r}}\right|^2 \\
e^{i\vec{k}\cdot\vec{r}} &= \sum_{\ell} (2\ell + 1) i^\ell j_\ell(kr) P_\ell(\cos\theta), \\
P_\ell(\cos\theta) &= \sqrt{\frac{4\pi}{2\ell + 1}} Y_{\ell,m=0}(\theta,\phi), \\
P_0(x) = 1,\ P_1(x) &= x,\ P_2(x) = (3x^2 - 1)/3, \\
f(\Omega) &\equiv \sum_\ell (2\ell + 1) e^{i\delta_\ell} \sin\delta_\ell \frac{1}{k} P_\ell(\cos\theta) \\
\psi_{\vec{k}}(\vec{r})|_{R\to\infty} &= e^{i\vec{k}\cdot\vec{r}} + \frac{e^{ikr}}{r} f(\Omega), \\
\frac{d\sigma}{d\Omega} &= |f(\Omega)|^2,\quad \sigma = \frac{4\pi}{k^2}\sum_\ell (2\ell + 1)\sin^2\delta_\ell,\quad \delta \approx -ak \\
L_\pm|\ell, m\rangle &= \sqrt{\ell(\ell + 1) - m(m \pm 1)}|\ell, m \pm 1\rangle, \\
\langle\tilde{\beta}, J, M|T^k_q|\beta, \ell, m_\ell\rangle &= \langle JM|k, q, \ell, m_\ell\rangle \frac{\langle\tilde{\beta}, J||T^{(k)}||\beta, \ell, J\rangle}{\sqrt{2J + 1}}, \\
n &= \frac{(2s + 1)}{(2\pi)^d}\int_{k<k_f} d^dk,\quad d\ \text{dimensions}, \\
\{\Psi_s(\vec{x}), \Psi^\dagger_{s'}(\vec{y})\} &= \delta^3(\vec{x} - \vec{y})\delta_{ss'}, \\
\Psi^\dagger_s(\vec{r}) &= \frac{1}{\sqrt{V}}\sum_{\vec{k}} e^{i\vec{k}\cdot\vec{r}} a^\dagger_s(\vec{k}),\quad \{\Psi_s(\vec{x}), a^\dagger_\alpha\} = \phi_{\alpha,s}(\vec{x}).
\end{aligned}
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1. (10 pts) Consider spin-vector operators, $\vec{L}$, $\vec{S}$ and $\vec{J} = \vec{L} + \vec{S}$. You can assume $\vec{S}$ operates on intrinsic spin and that $\vec{L}$ describes orbital angular momentum. Circle the operators that commute with J_z?

   - L_x
   - S_z ✓
   - J_x
   - $J_x^2 + J_y^2 + J_z^2$ ✓
   - $S_x^2 + S_y^2 + S_z^2$ ✓
   - $\vec{L} \cdot \vec{S}$ ✓

2. (5 pts) A particle of mass m moving in one-dimension feels the potential

$$V(x) = \begin{cases} \infty, & x < 0 \\ -V_0, & 0 < x < a \\ 0, & x > a \end{cases}$$

   Here V_0 is a positive constant. Assume that m, V_0 and a are initially chosen such that there exists one, but only one, bound state.

   Circle the actions that could increase the number of bound states

   - Increasing the magnitude of V_0 ✓
   - Increasing the distance a ✓
   - Increasing the mass m. ✓

3. (5 pts) A particle of mass m moving in one-dimension feels the potential

$$V(x) = \begin{cases} \infty, & x < 0 \\ -V_0\delta(x-a), & x > 0 \end{cases}$$

   Here V_0 is a positive constant. Assume that m, V_0 and a are initially chosen such that there exists one, but only one, bound state.

   Circle the actions that could increase the number of bound states

   - Increasing the magnitude of V_0
   - Increasing the distance a
   - Increasing the mass m.

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4. A proton and a neutron are in the ground state of a harmonic oscillator. An interaction is added,

$$V_{\text{s.s.}} = -\mu_n B S_{nz} - \mu_p B S_{pz},$$

where S_{nz} and S_{pz} are the spin projection operators for the proton and neutron. We will use $|JM\rangle$ to denote a state of total angular momentum J and projection M. At $t = 0$ one is in the state $|J = 0, M = 0\rangle$. Find the probability you are in the following states as a function of time.

(a) (5 pts) $|J = 0, M = 0\rangle$

(b) (5 pts) $|J = 1, M = 1\rangle$

(c) (5 pts) $|J = 1, M = 0\rangle$

(d) (5 pts) $|J = 1, M = -1\rangle$

Feel free to define the quantities

$$\omega_p \equiv -\mu_p B/2, \quad \omega_n \equiv -\mu_n B/2$$

to simplify your answers.

Solution:

$$\begin{aligned}
|\psi(t=0)\rangle &= \frac{1}{\sqrt{2}}\{|m_p = 1/2, m_n = -1/2\rangle - |m_p = -1/2, m_n = 1/2\rangle)\} \\
|\psi(t)\rangle &= \frac{1}{\sqrt{2}}\{e^{-i(\omega_p-\omega_n)t}|m_p = 1/2, m_n = -1/2\rangle \\
&\qquad -e^{-i(\omega_n-\omega_p)t}|m_p = -1/2, m_n = 1/2\rangle\}, \\
&= \frac{1}{2}\{e^{-i(\omega_p-\omega_n)t}(|J = 1, M = 0\rangle + |J = 0, M = 0\rangle) \\
&\qquad -e^{-i(\omega_n-\omega_p)t}(|J = 1, M = 0\rangle - |J = 0, M = 0\rangle),\} \\
&= \cos[(\omega_p - \omega_n)t]|J = 0, M = 0\rangle + i\sin[(\omega_p - \omega_n)t]|J = 1, M = 0\rangle.
\end{aligned}$$

a) $\cos^2(\omega_p - \omega_n)t$
b) Zero
c) $\sin^2(\omega_p - \omega_n)t$
d) Zero

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(Extra work space for #4)

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5. A beam of spinless particles of mass m and kinetic energy E is aimed at a spherically symmetric repulsive potential

$$V(r) = \begin{cases} V_0, & r < a \\ 0, & r > a \end{cases}$$

Assume $E < V_0$.

(a) (15 pts) Find the $\ell = 0$ phase shift as a function of the incoming wave number k.

(b) (10 pts) What is the cross section as $k \to 0$?

Solution:
a)

$$\begin{aligned}
\psi_I(r) &= A\sinh(qr), \\
\psi_I I(r) &= \sin(kr+\delta), \\
A\sinh(qa) &= \sin(ka+\delta), \\
qA\cosh(qa) &= k\cos(ka+\delta), \\
\frac{1}{q}\tanh(qa) &= \frac{1}{k}\tan(ka+\delta), \\
\delta &= -ka + \tan^{-1}\left(\frac{k}{q}\tanh(qa)\right), \\
q &= \sqrt{2m(V_0 - E_k)/\hbar^2}, \quad E_k = \frac{\hbar^2 k^2}{2m}.
\end{aligned}$$

b) As $k \to 0$,

$$\begin{aligned}
\delta &\approx -ka + \frac{k}{q_0}\tanh(q_0 a), \\
q_0 &= \sqrt{2mV_0/\hbar^2}, \\
\sigma &= \frac{4\pi}{k^2}\sin^2\delta, \\
\sigma(k=0) &= 4\pi\left(\frac{\tanh(q_0 a)}{q_0} - a\right)^2
\end{aligned}$$

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(Extra work space for #5)

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6. A particle of mass m moves in a one-dimensional potential

$$V(x) = \alpha|x|,$$

where V_0 is a positive constant. **NOTE THE ABSOLUTE VALUE!!!!**
Using a Gaussian form for a trial wave function,

$$\langle x|b\rangle = \psi_b(x) = \frac{1}{(\pi b^2)^{1/4}} e^{-x^2/2b^2},$$

where b is the variational parameter,

(a) (10 pts) What is $\langle b|KE|b\rangle$? –the expectation of the kinetic energy.

(b) (10 pts) What is $\langle b|V|b\rangle$? –the expectation of the potential energy.

(c) (15 pts) What is the estimate of the ground state energy?

Solution:
a)

$$\begin{aligned}
\partial_x^2|b\rangle &= \partial_x \frac{-x}{b^2}|b\rangle, \\
&= \left(\frac{-1}{b^2} + \frac{x^2}{b^4}\right)|b\rangle, \\
\langle b|x^2|b\rangle &= \frac{b^2}{2}, \quad \text{(you can show this different ways)} \\
\langle b|\frac{-\hbar^2}{2m}\partial_x^2|b\rangle &= \frac{\hbar^2}{4mb^2} = \langle b|\mathrm{KE}|b\rangle
\end{aligned}$$

b)

$$\begin{aligned}
\langle b|V|b\rangle &= \alpha \int_0^\infty dx \, \frac{1}{(\pi b^2)^{1/2}} 2x e^{-x^2/b^2}, \\
&= \frac{\alpha b}{\pi^{1/2}}.
\end{aligned}$$

c)

$$
\begin{aligned}
\mathbf{0} &= \partial_b \left(\frac{\hbar^2}{4mb^2} + \frac{\alpha b}{\pi^{1/2}} \right) \\
&= -\frac{\hbar^2}{2mb^3} + \frac{\alpha}{\pi^{1/2}}, \\
b^3 &= \frac{\hbar^2 \pi^{1/2}}{2m\alpha}, \\
E &> \frac{\hbar^2}{4m} \left(\frac{2m\alpha}{\hbar^2 \pi^{1/2}} \right)^{2/3} + \frac{\alpha}{\pi^{1/2}} \left(\frac{\hbar^2 \pi^{1/2}}{2m\alpha} \right)^{1/3} \\
&= \frac{\hbar^{2/3} \alpha^{2/3}}{m^{1/3}} \left(2^{-4/3} \pi^{-1/3} + 2^{-1/3} \pi^{-1/3} \right) \\
&= \frac{3\hbar^{2/3} \alpha^{2/3}}{2m^{1/3}} (2\pi)^{-1/3} \\
&= \frac{3}{2} \left[\frac{\hbar^2 \alpha^2}{2\pi m} \right]^{1/3}
\end{aligned}
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(Extra work space for #6)