FINAL EXAM,

PHYSICS 851, FALL 2019

SECRET STUDENT NUMBER:
STUDNUMBER

Noon Friday, December 11, until 5:00 PM Friday, December 18

This exam is worth 100 points.

$$\int_{-\infty}^{\infty} dx\; e^{-x^2/(2a^2)} = a\sqrt{2\pi},$$

$$H = i\hbar\partial_t,\;\; \vec{P} = -i\hbar\nabla,$$

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\;\; \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

$$U(t,-\infty) = 1 + \frac{-i}{\hbar}\int_{-\infty}^{t} dt'\; V(t')U(t',-\infty),$$

$$\langle x|x'\rangle = \delta(x-x'),\; \langle p|p'\rangle = \frac{1}{2\pi\hbar}\delta(p-p'),$$

$$|p\rangle = \int dx\; |x\rangle e^{ipx/\hbar},\;\; |x\rangle = \int \frac{dp}{2\pi\hbar}|p\rangle e^{-ipx/\hbar},$$

$$H = \frac{P^2}{2m} + \frac{1}{2}m\omega^2x^2 = \hbar\omega(a^\dagger a + 1/2),$$

$$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}X - i\sqrt{\frac{1}{2\hbar m\omega}}P,$$

$$\psi_0(x) = \frac{1}{(\pi b^2)^{1/4}}e^{-x^2/2b^2},\;\; b^2 = \frac{\hbar}{m\omega},$$

$$\rho(\vec{r},t) = \psi^*(\vec{r}_1,t_1)\psi(\vec{r}_2,t_2)$$

$$\vec{j}(\vec{r},t) = \frac{-i\hbar}{2m}(\psi^*(\vec{r},t)\nabla\psi(\vec{r},t) - (\nabla\psi^*(\vec{r},t))\psi(\vec{r},t)) - \frac{e\vec{A}}{mc}|\psi(\vec{r},t)|^2.$$

$$H = \frac{(\vec{P}-e\vec{A}/c)^2}{2m} + e\Phi,$$

$$\text{For } V = \beta\delta(x-y):\quad -\frac{\hbar^2}{2m}\left(\frac{\partial}{\partial x}\psi(x)|_{y+\epsilon} - \frac{\partial}{\partial x}\psi(x)|_{y-\epsilon}\right) = -\beta\psi(y),$$

$$\vec{E} = -\nabla\Phi - \frac{1}{c}\partial_t\vec{A},\;\; \vec{B} = \nabla\times\vec{A},$$

$$\omega_{\rm cyclotron} = \frac{eB}{mc},$$

$$e^{A+B} = e^Ae^Be^{-C/2},\;\; \text{if } [A,B]=C,\; \text{and } [C,A]=[C,B]=0,$$

$$Y_{0,0} = \frac{1}{\sqrt{4\pi}},\quad Y_{1,0} = \sqrt{\frac{3}{4\pi}}\cos\theta,\quad Y_{1,\pm1} = \mp\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\pm\phi},$$

$$Y_{2,0} = \sqrt{\frac{5}{16\pi}}(3\cos^2\theta - 1),\quad Y_{2,\pm1} = \mp\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta e^{\pm i\phi},$$

$$Y_{2,\pm2} = \sqrt{\frac{15}{32\pi}}\sin^2\theta e^{\pm2i\phi},\quad Y_{\ell -m}(\theta,\phi) = (-1)^mY^*_{\ell m}(\theta,\phi).$$

$$|N\rangle = |n\rangle - \sum_{m\neq n} |m\rangle \frac{1}{\epsilon_m - \epsilon_n} \langle m|V|n\rangle + \cdots$$

$$E_N = \epsilon_n + \langle n|V|n\rangle - \sum_{m\neq n} \frac{|\langle m|V|n\rangle|^2}{\epsilon_m - \epsilon_n}$$

$$j_0(x) = \frac{\sin x}{x},\ n_0(x) = -\frac{\cos x}{x},\ j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x},\ n_1(x) = -\frac{\cos x}{x^2} - \frac{\sin x}{x}$$

$$j_2(x) = \left(\frac{3}{x^3} - \frac{1}{x}\right)\sin x - \frac{3}{x^2}\cos x,\ n_2(x) = -\left(\frac{3}{x^3} - \frac{1}{x}\right)\cos x - \frac{3}{x^2}\sin x,$$

$$\frac{d}{dt}P_{i\to n}(t) = \frac{2\pi}{\hbar}|V_{ni}|^2\delta(E_n - E_i),$$

$$\frac{d\sigma}{d\Omega} = \frac{m^2}{4\pi^2\hbar^4}\left|\int d^3r\mathcal{V}(r)e^{i(\vec{k}_f - \vec{k}_i)\cdot\vec{r}}\right|^2,$$

$$\sigma = \frac{(2S_R+1)}{(2S_1+1)(2S_2+1)}\frac{4\pi}{k^2}\frac{(\hbar\Gamma_R/2)^2}{(\epsilon_k - \epsilon_r)^2 + (\hbar\Gamma_R/2)^2},$$

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_{\rm single}\tilde{S}(\vec{q}),\quad \tilde{S}(\vec{q}) = \left|\sum_{\vec{a}} e^{i\vec{q}\cdot\vec{a}}\right|^2,$$

$$e^{i\vec{k}\cdot\vec{r}} = \sum_\ell (2\ell+1)i^\ell j_\ell(kr)P_\ell(\cos\theta),$$

$$P_\ell(\cos\theta) = \sqrt{\frac{4\pi}{2\ell+1}}Y_{\ell,m=0}(\theta,\phi),$$

$$P_0(x) = 1,\ P_1(x) = x,\ P_2(x) = (3x^2-1)/3,$$

$$f(\Omega) \equiv \sum_\ell (2\ell+1)e^{i\delta_\ell}\sin\delta_\ell\frac{1}{k}P_\ell(\cos\theta)$$

$$\psi_{\vec{k}}(\vec{r})|_{R\to\infty} = e^{i\vec{k}\cdot\vec{r}} + \frac{e^{ikr}}{r}f(\Omega),$$

$$\frac{d\sigma}{d\Omega} = |f(\Omega)|^2,\quad \sigma = \frac{4\pi}{k^2}\sum_\ell (2\ell+1)\sin^2\delta_\ell,\quad \delta \approx -ak$$

$$L_\pm|\ell,m\rangle = \sqrt{\ell(\ell+1) - m(m\pm 1)}|\ell, m\pm 1\rangle,$$

$$C^{\ell,s}_{m_\ell,m_s;JM} = \langle \ell,s,J,M|\ell,s,m_\ell,m_s\rangle,$$

$$\langle\tilde{\beta},J,M|T^k_q|\beta,\ell,m_\ell\rangle = C^{k\ell}_{qm_\ell;JM}\frac{\langle\tilde{\beta},J||T^{(k)}||\beta,\ell,J\rangle}{\sqrt{2J+1}},$$

$$n = \frac{(2s+1)}{(2\pi)^d}\int_{k<k_f} d^dk,\quad d \text{ dimensions},$$

$$\{\Psi_s(\vec{x}),\Psi^\dagger_{s'}(\vec{y})\} = \delta^3(\vec{x}-\vec{y})\delta_{ss'},$$

$$\Psi^\dagger_s(\vec{r}) = \frac{1}{\sqrt{V}}\sum_{\vec{k}} e^{i\vec{k}\cdot\vec{r}}a^\dagger_s(\vec{k}),\quad \{\Psi_s(\vec{x}),a^\dagger_\alpha\} = \phi_{\alpha,s}(\vec{x}).$$

1. (5 pts) Consider three spin operators $\boldsymbol{S_x}, \boldsymbol{S_y}$ and $\boldsymbol{S_z}$. Circle the operators that commute with $\boldsymbol{S_z}$.
   - $\boldsymbol{S_x}$
   - $\boldsymbol{S_z}$
   - $\boldsymbol{S_x^2}$
   - $\boldsymbol{S_z^2}$
   - $\boldsymbol{S_x^2 + S_y^2 + S_z^2}$
2. (5 pts) Consider two sets of spin operators, $\boldsymbol{S_x, S_y, S_z}$ and $\boldsymbol{L_x, L_y, L_z}$. You can assume $\vec{\boldsymbol{S}}$ operates on intrinsic spin and that $\vec{\boldsymbol{L}}$ describes orbital angular momentum. Circle the operators that commute with $\boldsymbol{S_z}$.
   - $\boldsymbol{L_x}$
   - $\boldsymbol{L_z}$
   - $\boldsymbol{L_x^2}$
   - $\boldsymbol{L_z^2}$
   - $\boldsymbol{L_x^2 + L_y^2 + L_z^2}$
3. (5 pts) Now consider the operators $\vec{\boldsymbol{J}} \equiv \vec{\boldsymbol{L}} + \vec{\boldsymbol{S}}$. Circle the operators that commute with $\boldsymbol{S_z}$.
   - $\boldsymbol{J_x}$
   - $\boldsymbol{J_z}$
   - $\boldsymbol{J_x^2}$
   - $\boldsymbol{J_z^2}$
   - $\boldsymbol{J_x^2 + J_y^2 + J_z^2}$

4. (A proton and a neutron are in the ground state of a harmonic oscillator. An interaction is added,

$$V_{\rm s.s.} = -\alpha \vec{S}_p \cdot \vec{S}_n$$

At $t = 0$ the proton is in a $|\uparrow\rangle$ state and the neutron is in a $|\downarrow\rangle$ state, which we label as $|\uparrow,\downarrow\rangle$. With this labeling the first spin refers to the proton and the second to the neutron.

(a) (15 pts) In the basis above, express $V_{\rm s.s.}$ as a 4×4 matrix. Use a basis where the states are expressed as

$$|\uparrow,\uparrow\rangle = \begin{pmatrix}1\\0\\0\\0\end{pmatrix},\quad |\uparrow,\downarrow\rangle = \begin{pmatrix}0\\1\\0\\0\end{pmatrix},\quad |\downarrow,\uparrow\rangle = \begin{pmatrix}0\\0\\1\\0\end{pmatrix},\quad |\downarrow,\downarrow\rangle = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}.$$

(b) (15 pts) Find the probability that the pair is each of the following states as a function of time for $t > 0$.

i. $|\uparrow,\uparrow\rangle$
ii. $|\uparrow,\downarrow\rangle$ (this is the state at $t = 0$)
iii. $|\downarrow,\uparrow\rangle$
iv. $|\downarrow,\downarrow\rangle$

a) $V_{ss} = -\frac{\alpha}{2}\left[(\vec{S}_p+\vec{S}_n)^2 - |\vec{S}_p|^2 - |\vec{S}_n|^2\right]$

$= -\frac{\alpha}{2}\left[S(S+1) - \frac{3}{2}\right]\hbar^2$

$\begin{pmatrix}1\\0\\0\\0\end{pmatrix} = |S=1, M=1\rangle \qquad \begin{pmatrix}0\\0\\0\\1\end{pmatrix} = |S=1, M=-1\rangle$

$\begin{pmatrix}0\\1\\0\\0\end{pmatrix} = \frac{1}{\sqrt{2}}\left[|S=1, M=0\rangle + |S=0, M=0\rangle\right]$

$\begin{pmatrix}0\\0\\1\\0\end{pmatrix} = \frac{1}{\sqrt{2}}\left[|S=1, M=0\rangle - |S=0, M=0\rangle\right]$

$$V = \begin{pmatrix} -\frac{\alpha\hbar^2}{4} & 0 & 0 & 0 \\ 0 & \frac{\alpha\hbar^2}{4} & -\frac{\alpha\hbar^2}{2} & 0 \\ 0 & -\alpha\hbar^2/2 & \frac{\alpha\hbar^2}{4} & 0 \\ 0 & 0 & 0 & -\frac{1}{4}\alpha\hbar^2 \end{pmatrix}$$

(Extra work space for #4)

$\psi(t) = e^{-iV_{cs}t/\hbar}$

Look at central 2×2 sub matrix

$$\tilde{V} = \begin{pmatrix} \alpha\hbar^2/4 & -\alpha\hbar^2/2 \\ -\frac{\alpha\hbar^2}{2} & \alpha\hbar^2/4 \end{pmatrix} = \frac{\alpha\hbar^2}{4} - \frac{\alpha\hbar^2}{2}\sigma_x$$

$$e^{-iVt/\hbar} = e^{-i\frac{\alpha\hbar t}{4}} e^{-i\frac{\alpha\hbar t}{2}\sigma_x}$$

$$= e^{-i\frac{\alpha\hbar t}{4}}\left(\cos\frac{\alpha\hbar t}{2} - i\sigma_x \sin\frac{\alpha\hbar t}{2}\right)$$

$$\psi(t) = e^{-iVt/\hbar}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = e^{-i\alpha\hbar t/4}\begin{pmatrix} 0 \\ \cos\frac{\alpha\hbar t}{2} \\ -i\sin\frac{\alpha\hbar t}{2} \\ 0 \end{pmatrix}$$

$\text{Prob}(\uparrow\uparrow) = 0$

$\text{Prob}(\uparrow,\downarrow) = \cos^2\frac{\alpha\hbar t}{2}$

$\text{Prob}(\uparrow,\downarrow) = \sin^2\frac{\alpha\hbar t}{2}$

$\text{Prob}(\downarrow\downarrow) = 0$

5. A beam of spinless particles of mass m and kinetic energy E is aimed at a spherically symmetric repulsive potential

$$V(r) = \begin{cases} V_0, & r < a \\ 0, & r > a \end{cases}$$

Assume $E < V_0$.

(a) (10 pts) Find the $\ell = 0$ phase shift as a function of the incoming wave number k.

(b) (5 pts) What is the cross section as $k \to 0$?

(c) (10 pts) What is the relative probability density for a particle in the wave packet to be at the origin compared to the probability with no potential? I.e. If ρ_0 is the probability density at $r = 0$ in the absence of the potential and ρ is the density with the potential, find ρ/ρ_0.

a) $\psi_I = A \sinh q r$, $q = \sqrt{\frac{2mV_0}{\hbar^2} - k^2}$

$\psi_{II} = \sin kr + \delta$

B.C. $A \sinh qa = \sin(ka + \delta)$

$qA \cosh qa = k\cos(ka + \delta)$

$\frac{1}{q} \tanh qa = \frac{1}{k} \tan(ka + \delta)$

$\delta = -ka + \tan^{-1}\left\{\frac{k}{q} \tanh qa\right\}$

b) $\sigma = \frac{4\pi}{k^2} \sin^2\delta \simeq \frac{4\pi}{k^2}\left\{k\frac{\tanh q_0 a}{q_0} - ka\right\}^2$

$= 4\pi a^2 \left\{1 - \frac{\tanh q_0 a}{q_0 a}\right\}^2$

where $q_0 \equiv \sqrt{\frac{2mV_0}{\hbar^2}}$

c) $\rho = A^2 \frac{\sinh^2 qr}{\sin^2 kr}\bigg|_{r \to 0}$

$= A^2 \frac{q^2}{k^2} = \left(\frac{V_0}{E} - 1\right) \cdot A^2 = \left(\frac{V_0}{E} - 1\right) \frac{\sin^2(ka + \delta)}{\sinh^2 qa}$

(Extra work space for #5)

$$\sin^2(ka+\delta) = \sin^2\left\{\tan^{-1}\left[\frac{k}{q}\tanh qa\right]\right\}$$

$$= 1 - \frac{1}{1+\frac{k^2}{q^2}\tanh^2 qa}$$

$$= \frac{k^2/q^2 \tanh^2 qa}{1+\frac{k^2}{q^2}\tanh^2 qa}$$

$$= \frac{k^2\sinh^2 qa}{q^2\cosh^2 qa + k^2\sinh^2 qa} = \frac{\rho}{\rho_0}$$

6. A particle of mass m moves in a one-dimensional attractive potential

$$V(x) = -V_0 \exp(-x^2/2a^2).$$

Use a gaussian form for a trial wave function,

$$\langle x|b\rangle = \psi_b(x) = \frac{1}{(\pi b^2)^{1/4}} e^{-x^2/2b^2},$$

where b is the variational parameter.

(a) (10 pts) What is $\langle b|KE|b\rangle$? –the expectation of the kinetic energy.

(b) (10 pts) What is $\langle b|V|b\rangle$? –the expectation of the potential energy.

(c) (10 pts) Find an expression that when solved for b and then plugged into (a) and (b) provides an estimate of the energy. This expression can be a polynomial that needs to be solved for b. (No credit will be given for expressions that are dimensionally inconsistent)

(Extra work space for #6)

6) a) $\int e^{-x^2/2b^2} - \partial_x^2 \, e^{-x^2/2b^2} \, dx \quad 1/\sqrt{\pi b^2}$

$= \int e^{-x^2/2b^2} \frac{\partial}{\partial x} \frac{x}{b^2} e^{-x^2/2b^2} \quad 1/\sqrt{\pi b^2}$

$= \int e^{-x^2/b^2} \frac{x^2}{b^4} \frac{1}{\sqrt{\pi b^2}} \, dx$

$= \frac{1}{2b^2} \qquad \langle KE \rangle = \frac{\hbar^2}{4 m b^2}$

b) $\langle V \rangle = V_0 \int e^{-\frac{x^2}{b^2} - \frac{x^2}{2a^2}} dx \, \frac{1}{\sqrt{\pi b^2}}$

$\frac{1}{c^2} = \frac{1}{b^2} + \frac{1}{2a^2}, \quad c^2 = \frac{2a^2 b^2}{2a^2 + b^2}$

$\langle V \rangle = -V_0 \frac{c}{b}$

$= -\sqrt{2} V_0 \frac{a}{\sqrt{2a^2 + b^2}}$

c) $\frac{\partial V}{\partial b} = +\sqrt{2} V_0 \frac{ab}{(2a^2 + b^2)^{3/2}} \qquad \frac{\partial KE}{\partial b} = -\frac{\hbar^2}{2 m b^3}$

$\frac{2 V_0^2 a^2 b^2}{(2a^2 + b^2)^3} = \frac{\hbar^4}{4 m^2 b^6}, \quad \frac{b^2}{(2a^2 + b^2)^3} = \frac{\alpha a^2}{b^6}$

$\alpha \equiv \frac{\hbar^4}{8 m^2 a^4} \frac{1}{V_0^2}$

$$\frac{b^2}{(2a^2+b^2)^3} = \frac{\alpha a^2}{b^6}, \quad b^8 - \alpha a^2 (2a^2+b^2)^3 = 0$$

$$A_4 b^8 + A_3 b^6 + A_2 b^4 + A_1 b^2 + A_0 = 0$$

$$A_4 = 1, \quad A_3 = -\alpha a^2, \quad A_2 = -6a^4 \alpha$$

$$A_1 = -12 a^6 \alpha, \quad A_0 = -8a^8 \alpha$$

$$\alpha \equiv \frac{\hbar^4}{8m^2a^4} \frac{1}{V_0^2}$$

Quartic equation