Electrodynamics Subject Exam Prep Quiz, Friday, April 21, 2017

your name(s)____________________________________

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E & M Study Guide

Ch 1 Special Relativity Primer

1.1 Gamma Factors and Such

- $t_{lab} = \gamma t_0$
- $L_{lab} = \frac{L_0}{\gamma}$

1.2 Lorentz Transformations

- $r^\alpha = L^\alpha_\beta r'^\beta$
- $$L^\alpha_\beta = \begin{bmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$ for $\hat{x}$ boost $\gamma^2(1-v^2) = 1$
- $sinh\eta \equiv \gamma v$, $cosh\eta \equiv \gamma$, $\eta =$ "rapidity"

1.3 invariants and the Metric Tensor $g^{\alpha\beta}$

- Dot product: $A^\alpha g_{\alpha\beta} B^\beta = A^\alpha B_\alpha =$ invariant
- $$g_{\alpha\beta} = g^{\alpha\beta} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$
- $\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$
- Continuity Eqn: $\partial \cdot J = 0$
- Maxwell's Eqns:
  $\partial_\alpha F^{\alpha\beta} = J^\beta$
  $\partial_\alpha \tilde{F}^{\alpha\beta} = 0$

1.4 Four-Velocities and Momenta

- $u \equiv (\gamma, \gamma \frac{d\vec{x}}{dt}$
- $u^\alpha u_\alpha = 1$
- $p \equiv mu$
- $p^\alpha p_\alpha = m^2$

Ch 2 Dynamics of a Relativistic Point

2.1 Lagrangian for a Free Relativistic Particle

- $S = -m \int dt \sqrt{(\frac{dt'}{dt})^2 - (\frac{d\vec{r}'}{dt})^2}$, $r'^\alpha = r^\alpha + \delta\Omega^{\alpha\beta}(t) r_\beta$
- $\delta S = \int dt \frac{\partial \mathcal{L}}{\partial r^\alpha} \frac{d}{dt}(\delta\Omega^{\alpha\beta} r_\beta)$
  $= \int dt[-\frac{d}{dt}(\pi_\alpha r_\beta) + \pi_\alpha \dot{r}_\beta]\delta\Omega^{\alpha\beta}$
  $= 0$
- $\frac{d}{dt}(r^\alpha p^\beta - p^\alpha r^\beta) = 0$

2.2 Interaction of a Charged Particle with an External EM Field

- $S = \int d\tau(-m - eu \cdot A)$
  $= -m \int dt \sqrt{(\frac{dt'}{dt})^2 - (\frac{d\vec{r}'}{dt})^2} - e \int dt (A_0 - \vec{v} \cdot \vec{A})$
- $\mathcal{L} = -m\sqrt{1-v^2} - e((A_0 - \vec{v} \cdot \vec{A})$
- $\frac{d}{dt}(\gamma m v_i) = -e\partial_i A_0 - e\partial_t A_i + e\vec{v} \times (\vec{\Delta} \times \vec{A})$
- $\vec{E} = -\vec{\Delta} A_0 - \partial_t \vec{A}$
- $\vec{B} = \vec{\Delta} \times \vec{A}$
- $\frac{d}{dt}(\gamma m \vec{v}) = e\vec{E} + e\vec{v} \times \vec{B}$

2.3 Motion in a Constant Magentic Field

- $\vec{A} = xB\hat{y}$, $\vec{B} = B\hat{z}$
- $S = -\int dt[m\sqrt{1-v^2} - exBv_y]$
- $\frac{d}{dt}(\gamma m \vec{v}) = e(v_y \hat{x} - v_x \hat{y})B$
- $\dot{v}_x = \frac{eB}{\gamma m} v_y = \omega v_y$
- $\dot{v}_y = -\frac{eB}{\gamma m} v_x = -\omega v_x$
- $v_x = \omega R sin(\omega t)$, $x = R cos(\omega t)$
- $v_y = -\omega R cos(\omega t)$, $y = R sin(\omega t)$

Gauge Transformations

- $A'^\mu = A^\mu + \partial^\mu \Lambda(t, x, y, z)$
- $\vec{E}, \vec{B}$ do not change

2.5 The EM Field Tensor

- $F^{\alpha\beta} \equiv \partial^\alpha A^\beta - \partial^\beta A^\alpha$

$$F_{\alpha\beta} = \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{bmatrix}$$

$$F^{\alpha\beta} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix}$$

$$F^\alpha_\beta \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{bmatrix}$$

- $\frac{d}{d\tau} u^\alpha = eF^{\alpha\beta} u_\beta$

Ch 3 Dynamics of EM Fields

3.1 Lagrangian (Density) for Free Fields: Deriving Maxwell's Equations

- $S_f = \int d^4 r \mathcal{L}(a^\mu, \partial_\nu A^\mu)$
- $\partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu A^\nu} = \frac{\partial \mathcal{L}}{\partial A^\nu}$
- $\mathcal{L} = -\frac{1}{16\pi} F^{\mu\nu} F_{\mu\nu} = \frac{1}{16\pi} F^{\mu\nu} F_{\nu\mu}$
- $S_m = \int d^4 r J \cdot A$
- $\partial_\mu F^{\mu\nu} = 4\pi J^\nu$
- $J^\alpha \equiv \frac{1}{\Omega} \sum_{a \in \Omega} q_a \frac{u_a^\alpha}{\gamma_a}$, Ω is 4D volume
- $\vec{\Delta} \cdot \vec{E} = 4\pi J^0$
- $(\vec{\Delta} \times \vec{B}) - \partial_t \vec{E} = 4\pi \vec{J}$
- Dual EM tensor

$$\tilde{F}^{\alpha\beta} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta} = \begin{bmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{bmatrix}$$

- $\partial_\mu \tilde{F}^{\mu\nu} = 0$
- $\vec{\Delta} \cdot \vec{B} = 0$
- $\partial_t \vec{B} + \vec{\Delta} \times \vec{E} = 0$

3.2 Pseudo-Vectors and Pseudo-Scalers

- $F^{\alpha\beta} \tilde{F}_{\alpha\beta} = -4\vec{E} \cdot \vec{B}$, pseudoscalar
- $F^{\alpha\beta} F_{\alpha\beta} = -2(|\vec{E}|^2 - |\vec{B}|^2)$, (regular) scalar

The Stress-Energy Tensor of the EM Field

- $T^{\alpha\beta} \equiv pi^\alpha \partial^\beta \phi - g^{\alpha\gamma} \mathcal{L}$, $\phi = A^\mu$, $\pi^\alpha = \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi)}$
- $T^{\alpha\gamma} = \frac{1}{4\pi} F^{\alpha\beta} F_\beta^{\ \gamma} + F^{\mu\nu} F_{\mu\nu}$
- $T^{00} = \frac{1}{8\pi}(E^2 + B^2)$
- $T^{0i} = \frac{1}{4\pi}\epsilon_{ijk}(E_j B_k)$
- $T^{ij} = \frac{1}{8\pi}(\delta_{ij}(E^2 + B^2) - 2E_i E_j - 2B_i B_j)$
- $U^{(f)} = \frac{1}{2} \int d^3 r A_0(\vec{r}) J_0(\vec{r})$

3.4 Hyper-Surfaces and Conservation of $E, \vec{P}, \vec{L}$

- stress-energy tensor is symmetric

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LONG ANSWER SECTION

1. Consider a region with non-zero electric and magnetic fields,

$$\vec{B} = B_z\hat{z}, \quad \vec{E} = E_x\hat{x},$$

where $|B_z| >> |E_x|$.

(a) (2 pts) Is there is a reference frame where $\vec{B} = \mathbf{0}$.

(b) (2 pts) Is there is a reference frame where $\vec{E} = \mathbf{0}$.

(c) (16pts) If a particle of mass m and charge e begins at rest at the origin, find the velocity as a function of time. (Note: You may want to consider whether you need to consider whether the motion is relativistic)

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Extra workspace for #1

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2. Cathy Couchsitter, whose momentum is $\boldsymbol{P_c} = \boldsymbol{(M_c, 0, 0, 0)}$, observes two events at coordinates described by $\boldsymbol{x_a}$ and $\boldsymbol{x_b}$. Cathy observes Randy Rabbit, whose mass is $\boldsymbol{M_r}$ zipping by with momentum $\boldsymbol{P_r}$.

   (a) (5 pts) According to Cathy, what is the separation in time of the two events?

   (b) (5 pts) According to Randy, what is the separation in time of the two events?

   Express both answers in terms of Lorentz invariants using $\boldsymbol{M_c}$, $\boldsymbol{M_r}$, $\boldsymbol{P_c}$, $\boldsymbol{P_r}$, $\boldsymbol{x_a}$ and $\boldsymbol{x_b}$.

Extra workspace for #2

your name(s)____

3. Consider two very large parallel capacitor plates of area $\boldsymbol{A}$, carrying charge densities $\boldsymbol{\sigma}$ and $-\boldsymbol{\sigma}$, and oriented perpendicular to the $\boldsymbol{z}$ axis. The plates are initially at a very small separation at $\boldsymbol{t = 0}$, but are pulled apart, moving with constant non-relativistic velocities $\boldsymbol{v/2}$ and $\boldsymbol{-v/2}$.

   (a) (5 pts) What is the electric field between the plates?

   (b) (5 pts) Find all four non-zero elements of the stress-energy tensor ($\boldsymbol{T_{\alpha\beta}}$). Check that the stress-energy tensor is traceless.

   (c) (5 pts) What is the power (energy per time) at which the field energy between the plates increases due to the growing volume? Ignore fringe effects.

your name(s)____

Extra work space for #3

your name(s)____

4. TRUE OR FALSE (2 pts each)

   (a) For electromagnetic fields at a given point x, you can always find a reference frame where either $\vec{E}(x) = 0$ or $\vec{B}(x) = 0$. _______

   (b) For two events at space-time points x_a and x_b, if one observer records a occurring before b, then all observers must observe a before b. _______

   (c) Under parity transformation $\vec{E} \cdot \vec{B}$ is unchanged.

   (d) A nucleus is spin-polarized by being placed in a strong magnetic field $\vec{B} = B\hat{z}$. The nucleus then undergoes a decay where neutrons, which are products of the decay, are observed to be emitted more often in the $+\hat{z}$ direction than in the $-\hat{z}$ direction. This is evidence of parity violation. _______

5. (8 pts)A charged particle moves through fields given by a vector potential,

$$A_0 = -Ex, \quad A_y = Bz.$$

Which of the following are conserved?

   (a) L_x ($\vec{L}$ is the orbital angular momentum)

   (b) L_y

   (c) L_z

   (d) $|\vec{L}|^2$

   (e) p_x

   (f) p_y

   (g) p_z

   (h) kinetic energy