PHY 841: Student Composed Questions

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Problem 1 Angular Distribution of Radiation from Relativistic Particles

Suppose you have a linear accelerator in which an electron with velocity $\beta = v/c$ is being accelerated.
a) W.r.t. the direction of $\vec{v}$, find the angle θ_{max} at which the maximum radiation is emitted.
b) Show that for the ultra-relativistic case ($\beta \to 1$), $\theta_{max} \approx \sqrt{\frac{1-\beta}{2}}$
c) Show that for the non-relativistic case ($\beta \to 0$), $\theta_{max} \approx \pi/2$

a)
For a linearly accelerated electron, the radiation power per solid angle is

$$\frac{dP}{d\Omega} = \frac{e^2}{4\pi(1-\beta\cos\theta)^5}|\dot{\vec{\beta}}|^2 \sin^2\theta \tag{1}$$

So the angle of maximum power is given by

$$\frac{d}{d\theta}\frac{dP}{d\Omega} = \frac{e^2|\dot{\vec{\beta}}|^2}{4\pi}\left[\frac{2\sin\theta\cos\theta}{(1-\beta\cos\theta)^5} - \frac{5\beta\sin^3\theta}{(1-\beta\cos\theta)^6}\right] = 0 \tag{2}$$

$\theta = 0$ corresponds to the angle of minimum power. So the angle of maximum power is given by

$$\frac{2\cos\theta}{(1-\beta\cos\theta)^5} - \frac{5\beta\sin^2\theta}{(1-\beta\cos\theta)^6} = 0 \tag{3}$$

i.e.

$$2\cos\theta(1-\beta\cos\theta) - 5\beta(1-\cos^2\theta) = 0 \tag{4}$$

Therefore,

$$\theta_{max} = \arccos\left(\frac{\sqrt{15\beta^2+1}-1}{3\beta}\right) \tag{5}$$

b)
Let $x = 1 - \beta$. When $\beta \to 1$, $x \to 0$. Expand $\cos\theta_{max}$ to the 1st order of x.

$$\begin{aligned}
\cos\theta_{max} =& \frac{\sqrt{15\beta^2+1}-1}{3\beta} \\
=& \frac{\sqrt{15(1-x)^2+1}-1}{3(1-x)} \\
=& \frac{1}{3}\left(4\sqrt{1-\frac{15}{8}x+\frac{15}{16}x^2}-1\right)(1-x)^{-1} \\
\approx& \frac{1}{3}\left[4\left(1-\frac{15}{16}x\right)-1\right](1+x) \\
=& \left(1-\frac{5}{4}x\right)(1+x) \\
\approx& 1-\frac{1}{4}x
\end{aligned} \tag{6}$$

So when $x \to 0$, $\cos\theta_{max} \to 1$. Then θ_{max} is very small. Thus,

$$\cos\theta_{max} \approx 1 - \frac{1}{2}\theta_{max}^2 \tag{7}$$

Then

$$1 - \frac{1}{2}\theta_{max}^2 \approx 1 - \frac{1}{4}x \Rightarrow \theta_{max} \approx \sqrt{\frac{x}{2}} = \sqrt{\frac{1-\beta}{2}} \tag{8}$$

c)
When $\beta \to 0$,

$$\frac{dP}{d\Omega} \approx \frac{e^2}{4\pi}|\dot{\vec{\beta}}|^2 \sin^2\theta \tag{9}$$

So the angle at which the maximum radiation is emitted is $\theta_{max} = \pi/2$
This result can also be obtained by

$$\lim_{\beta\to 0} \frac{\sqrt{15\beta^2+1}-1}{3\beta} = \lim_{\beta\to 0} \frac{\frac{30\beta}{\sqrt{15\beta^2+1}}}{3} = 0 \tag{10}$$

$$\Rightarrow \lim_{\beta\to 0}\theta_{max} = \lim_{\beta\to 0} \arccos\frac{\sqrt{15\beta^2+1}-1}{3\beta} = \arccos 0 = \frac{\pi}{2} \tag{11}$$