There is a source that is emitting a light, receding from an observer. The light is emitted at frequency f_s and with wavelength λ_s. It is received by the observer at f_o and with wavelength λ_o.

a. Derive the ratio between the frequency at the source to the frequency at the observer, with you answer in terms of β (or v if you like).

b. What is the redshift $\left(z = \frac{\lambda_o - \lambda_s}{\lambda_s}\right)$ in terms of β.

ANSWERS:

a. Label the events:
Event according to source $(\tau, 0)$ and Event according to observer $(\gamma\tau, \gamma v\tau)$.
Note: You can get these from $t' = \gamma(t - vx/c^2)$ and $x' = \gamma(x - vt)$.
Because of the observer's position when the event occurs, it will take $\gamma v\tau/c$ to reach them. Therefore,

$$\tau' = \gamma\tau + \gamma v\tau/c \tag{1}$$

$$t = \frac{\lambda_s}{c - v} = \frac{c/f_s}{c - v} = \frac{1}{(1 - \beta)f_s} \tag{2}$$

Time contraction, so $t_o = t/\gamma = t\sqrt{1 - \beta^2}$.

We can now use t_o to get f_o:

$$f_o = 1/t_o = \gamma/t \tag{3}$$

$$f_o = \gamma(1 - \beta)f_s \tag{4}$$

$$f_o = \frac{1}{\sqrt{1 - \beta^2}}(1 - \beta)f_s \tag{5}$$

$$f_o = \frac{(1 - \beta)}{\sqrt{(1 - \beta)(1 + \beta)}}f_s \tag{6}$$

$$f_o = \sqrt{\frac{1 - \beta}{1 + \beta}}f_s \tag{7}$$

$$\frac{f_o}{f_s} = \frac{1 - \beta}{1 + \beta} \tag{8}$$

$$\boxed{\frac{f_s}{f_o} = \frac{1 + \beta}{1 - \beta}} \tag{9}$$

b.

$$z = \frac{\lambda_o - \lambda_s}{\lambda_s} = \frac{\lambda_o}{\lambda_s} - \frac{\lambda_s}{\lambda_s} = \frac{f_s}{f_o} - 1 = \boxed{\sqrt{\frac{1 + \beta}{1 - \beta}} - 1} \tag{10}$$