Electrodynamics I
Chapter 4 Review Problem

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April 23, 2017

Consider a spherical shell with radius R which has potential $V(\theta, \phi) = V_0 \sin^2(\theta)\cos(2\phi)$. Find the potential everywhere.

General Solution

The potential outside the shell obeys Laplace's equation:

$$\nabla^2 \Phi = 0$$

The general solution in spherical coordinates is

$$\Phi(r, \theta, \phi) = \sum_{lm} \left[A_{lm} \left(\frac{r}{R}\right)^l + B_{lm} \left(\frac{r}{R}\right)^{-(l+1)} \right] Y_l^m(\theta, \phi)$$

We solve for the coefficients, A_{lm}, B_{lm}, using the orthonormality of the set of Y_l^m:

$$\int Y_{l'}^{m'*} Y_l^m \, d\Omega = \delta_{ll'}\delta_{mm'}$$

So we apply this to our boundary condition:

$$\Phi(R, \theta, \phi) = V(\theta, \phi) = \sum_{lm} [A_{lm} + B_{lm}] \, Y_l^m(\theta, \phi)$$

$$\int Y_{l'}^{m'*} \Big[\sum_{lm} [A_{lm} + B_{lm}] \, Y_l^m \Big] \, d\Omega = \int Y_{l'}^{m'*} V \, d\Omega$$

$$A_{lm} + B_{lm} = \int Y_l^{m*} V \, d\Omega$$

Now, in our case, we are constrained by the fact that our potential should be finite everywhere. Therefore, within the shell, $B_{lm} = 0$, and outside of the shell, $A_{lm} = 0$.

Now we use this method to decompose our boundary condition into the spherical harmonics:

$$A_{lm} + B_{lm} = \int Y_l^{m*} V_0 \sin^2\theta \cos 2\phi \, d\Omega$$

And we find that

$$A_{lm} + B_{lm} = \begin{cases} \frac{2\sqrt{2\pi}}{\sqrt{15}} V_0 & l = 2, m = \pm 2 \\ 0 & \text{otherwise} \end{cases}$$

So that the solution is

$$\Phi(r, \theta, \phi) = \begin{cases} \frac{2\sqrt{2\pi}}{\sqrt{15}} V_0 \left(\frac{r}{R}\right)^2 \left(Y_2^{-2}(\theta, \phi) + Y_2^2(\theta, \phi)\right) & r \le R \\ \frac{2\sqrt{2\pi}}{\sqrt{15}} V_0 \left(\frac{R}{r}\right)^3 \left(Y_2^{-2}(\theta, \phi) + Y_2^2(\theta, \phi)\right) & r > R \end{cases}$$

Azimuthal Symmetry

When the boundary conditions do not depend upon the azimuthal angle ($\frac{\partial V}{\partial \phi} = 0$), the set of functions which are solutions to the Laplace equations becomes

$$\Phi(r,\theta,\phi) = \sum_l \left[A_l \left(\frac{r}{R}\right)^l + B_l \left(\frac{r}{R}\right)^{-(l+1)} \right] P_l(\cos\theta)$$

Where P_l are the Legendre polynomials, which follow the orthogonality condition:

$$\int_{-1}^{1} P_m(x) P_n(x)\, dx = \frac{2}{2n+1}\delta_{mn}$$

As above, we apply this to our boundary condition:

$$\Phi(R,\theta,\phi) = V(\theta,\phi) = \sum_l \left[A_l + B_l\right] P_l(\cos\theta)$$

$$\int_0^{\pi} P_n(\cos\theta) \Big[\sum_l \left[A_l + B_l\right] P_l(\cos\theta) \Big] \sin\theta\, d\theta = \int P_l(\cos\theta) V \sin\theta\, d\theta$$

$$(A_l + B_l)\frac{2}{2l+1} = \int P_l(\cos\theta) V \sin\theta\, d\theta$$

$$A_l + B_l = \frac{2l+1}{2} \int P_l(\cos\theta) V \sin\theta\, d\theta$$

And using similar arguments as above, the relative values of A_l and B_l are constrained by the asymptotic behavior at $r = 0$ and $r = \infty$.

Now, take the boundary condition $V(\theta,\phi) = V_0 \sin^2\theta$. Applying this method, we find that

$$A_l + B_l = \begin{cases} \frac{2}{3}V_0 & l = 0 \\ -\frac{2}{3}V_0 & l = 2 \\ 0 & \text{otherwise} \end{cases}$$

So that the exact solution is

$$\Phi(r,\theta,\phi) = \begin{cases} \frac{2}{3}V_0 \left(P_0(\cos\theta) - \left(\frac{r}{R}\right)^2 P_2(\cos\theta) \right) & r \le R \\ \frac{2}{3}V_0 \left(\frac{R}{r} P_0(\cos\theta) - \left(\frac{R}{r}\right)^3 P_2(\cos\theta) \right) & r > R \end{cases}$$