LECTURE NOTES ON ELECTRODYNAMICS

PHY 841 – 2017

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These notes are for the one-semester graduate level electrodynamics course taught at Michigan State University. These notes are more terse than a text book, they do cover all the material used in PHY 841. They are **NOT** meant to serve as a replacement for a text. The course makes use of two textbooks: The Classical Theory of Fields by L.D. Landau and E.M. Lifshitz, and Classical Electrodynamics by J.D. Jackson. Anybody is welcome to use the notes to their heart's content, though the text should be treated with the usual academic respect when it comes to copying material. If anyone is interested in the LaTeX source files, they should contact me (prattsc@msu.edu). Solutions to the end-of-chapter problems are also provided on the course web site (http://www.pa.msu.edu/courses/phy841). Please beware that this is a web manuscript, and is thus alive and subject to change at any time.

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1 Special Relativity Primer

Electromagnetism is inherently relativistic. To see this, consider a charged particle moving through a magnetic field in deep space. The particle undergoes an acceleration proportional to its velocity because magnetic force, $\vec{F} = q\vec{v} \times \vec{R}$, depend on velocity. But, who defines the velocity? If an observer moves with the particle's velocity, the speed of the particle is zero, and therefore there is no magnetic force and no acceleration. Clearly, the acceleration cannot exist in one reference frame and disappear in another. The solution to this paradox is that if one boosts the observer to the frame of the particle, the boosted observer will then see an electric field. The fact that velocity boosts mix magnetic and electric field is related to the fact that space and time get interchanged in boosts due to the special theory of relativity. Thus, magnetism cannot be understood without relativity. The first part of the course will cover the relativistic formulation of Maxwell's Equations. All such material will seem intimidating without a firm grasp of the principles and formalism of the special theory of relativity in the context of classical physics.

Students should also read through the first chapter of Landau and Lifshitz.

1.1 γ Factors and Such

First, we review the standard arguments for Lorentz length contraction and for time dilation, i.e., we will demonstrate how a meter stick moving with velocity v appears shorter by a factor $1/\gamma$, where $\gamma = 1/\sqrt{1-(v/c)^2}$, and will also discuss how a moving clock has ticks separated by extended times, γ.

Both these results stem from the basic postulate of special relativity, that the speed of light is the same in all reference frames. First, we consider time dilation. Consider two mirrors separated by distance L_0. To an observer in the frame of the mirror, the time for light to bounce back and forth from the mirrors is $t_0 = 2L_0/c$. Now let the mirror move perpendicular to the path of the light with speed v. An observer in the laboratory sees the same light pulse move with speed c, but that the path is longer than $2L_0$ because the return point will have moved a distance $vt_{\rm lab}$, making the entire distance equal to

$$L = 2\sqrt{L_0^2 + (vt_{\rm lab}/2)^2}. \tag{1.1}$$

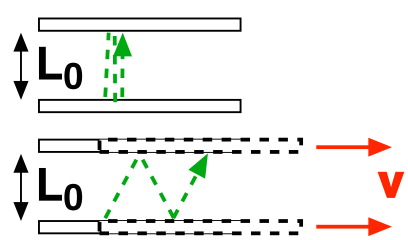

Figure 1.1: The upper clock functions by counting clicks separated in time by $t_0 = 2L_0/c$. If the clock is moving, but if the speed of light is the same, more time is required for the time between pulses because the distance traveled is larger.

The time for the return in the lab frame is thus

$$t_{\rm lab} = \frac{2\sqrt{L_0^2 + (vt_{\rm lab}/2)^2}}{c}. \tag{1.2}$$

Replacing L_0 with $ct_0/2$, one can solve for $t_{\rm lab}$ in terms of t_0,

$$t_{\rm lab} = \gamma t_0, \quad \gamma \equiv \frac{1}{\sqrt{1-(v/c)^2}}. \tag{1.3}$$

Thus, all moving clocks run slow. The most basic manifestation of this considers the lifetime of radioactive decays, which are extended by the factor γ. Note that for the future, we will often work in units where $c = 1$ to save ink and eyestrain.

Secondly, we consider the same set of mirrors, but instead let the apparatus move parallel to the path of the light (perpendicular to the plane of the mirrors). Again, the observer moving with the stick sees the time between pulses as $2L_0/c$. The laboratory observer measures,

$$t_{\rm lab} = \frac{L_{\rm lab}}{c+v} + \frac{L_{\rm lab}}{c-v} = \frac{2cL_{\rm lab}}{c^2-v^2} = \frac{2\gamma^2 L_{\rm lab}}{c}, \tag{1.4}$$

where here we have allowed the length of the meter stick to change from the value observed in the frame of the stick. Using the fact that $t_{\rm lab} = \gamma t_0 = 2\gamma L_0/c$,

$$\begin{aligned} 2\gamma\frac{L_0}{c} &= 2\gamma^2\frac{L_{\rm lab}}{c}, \\ L_{\rm lab} &= \frac{L_0}{\gamma}. \end{aligned} \tag{1.5}$$

Thus moving meter sticks appear shorter.

One has to be careful to note that the two simple expressions apply for very specific circumstances. The expression for time dilation, $t_{\rm lab} = \gamma t_0$, only applies when the time separates two events which occur at the same location in the frame of the moving observer measuring t_0. Also, the expression for length contraction, $L_{\rm lab} = L_0/\gamma$, is applicable only when the distance between the moving ends are measured simultaneously in time according to the observer measuring L_0. These subtleties are illustrated by the ladder paradox. Imagine a runner moving at 60% of the speed of light carrying a 10 foot ladder and moving through a 10 foot garage that has doors at both ends. The gamma factor is 1.25. Thus, an observer in the frame of the garage sees an 8-foot ladder moving through the garage, and could in principle close the garage at both ends trapping the ladder completely inside. The runner sees an 8-foot garage, and believes there was a moment when the front of the ladder had gone completely through the garage while the back of the ladder had not penetrated the front door. The paradoxical question is "Did the ladder fit?". The answer has to do with the simultaneity of two events. In this case it could be lights flashing at the front and back of the ladder. If the runner thinks the lights flash simultaneously, indeed both lights flash outside the garage and on opposite sides. The observer in the garage frame agrees with the assessment that the lights flashed outside the garage, but instead thinks that the light at the back of the ladder blinked first and that the light at the front of the ladder blinked later. Conversely, if the lights blinked such that the observer in the garage frame thought they were simultaneous, the two lights could have both flashed inside the garage. However, the runner would have recorded the light at the front of the ladder blinking first.

The ladder paradox underscores the importance of thinking of times and distances as describing the difference between two events, i.e., times and displacements are always relative to something. There are other famous paradoxes in relativity that also lead to a better understanding of the essence of the theory, such as the twin paradox or whether the radius of a rotating wheel contracts. However, the latter two involve acceleration and are thus related to the general theory of relativity, which is not considered here, but would be considered in a course on gravity.

1.2 Lorentz Transformations

Space and time are even mixed together in non-relativistic (Newtonian) transformations. For instance, consider an event that occurs at time t and position x in the laboratory frame (Here we consider only one spatial dimension). In a moving frame, the event occurs at:

$$x_{\rm lab} = x_0 + vt_0, \quad t_{\rm lab} = t_0, \tag{1.6}$$

in a Newtonian transformation. For a relativistic transformation, we assume a more general linear form,

$$\begin{aligned} x_{\rm lab} &= Ax_0 + Bvt_0, & (1.7)\\ t_{\rm lab} &= Ct_0 + Dvx_0, & (1.8) \end{aligned}$$

where A, B, C and D are functions of v^2. The powers of v are required by parity considerations. The inverse transformation should look the same, but with $v \rightarrow -v$,

$$\begin{aligned} x_0 &= Ax_{\rm lab} - Bvt_{\rm lab}, \\ t_0 &= Ct_{\rm lab} - Dvx_{\rm lab}. \end{aligned} \tag{1.9}$$

First, we consider the decay of a relativistic particle which passes by the point ($x = t = 0$) in both frames. Since the particle does not move in the particle frame ($x_0 = 0$). The transformations becomes

$$\begin{aligned} t_{\rm lab} &= Ct_0, \\ x_{\rm lab} &= Bvt_0. \end{aligned} \tag{1.10}$$

The expression for time dilation in the previous section then yields,

$$C = \gamma. \tag{1.11}$$

The decay occurs at the position $x_{\rm lab} = vt_{\rm lab}$, and since $t_{\rm lab} = \gamma t_0$, one finds

$$B = \gamma. \tag{1.12}$$

To solve for the other two coefficients, consider the ladder paradox. Assume a light at the back end of the latter blinks at a time $x = t = 0$, and at the front end of the ladder a light blinks simultaneously (in the frame of the ladder) at the space-time point, $x_0 = L_0, t_0 = 0$. In the laboratory frame the lights blink at times different by an amount DvL_0. The position at which the light blinks is

$$x_{\rm lab} = L_{\rm lab} + vt_{\rm lab}, \tag{1.13}$$

where $L_{\rm lab}$ is the length of the ladder. Since the apparent length of the ladder is shrunk by a factor γ, $L_{.\rm lab} = x_0/\gamma$, and

$$x_{\rm lab} = x_0/\gamma + vt_{\rm lab}, \tag{1.14}$$

Rearranged,

$$x_0 = \gamma x_{\rm lab} - \gamma vt_{\rm lab}. \tag{1.15}$$

This gives $A = \gamma$. To solve for D take the expressions, Eq.s (1.7,1.9,1.10) in the Lorentz transformations, and solve for D. One finds $D = \gamma$.

The expressions for the coefficients defined in Eq.s (1.7-1.10)) can be summed up as a matrix equation,

$$r^\alpha = L^\alpha_{\ \beta} r'^\beta, \tag{1.16}$$

$$\mathcal{L} = \left(\begin{array}{cc} \gamma & \gamma v \\ \gamma v & \gamma \end{array}\right),$$

Here, the indices α are either 0 or 1, with "0" referring to time component and "1" referring to the 'x" component. If we included y and z, there would be four components representing the coordinate of an event, t, x, y, z. These four components make up a "four-vector". The Lorentz matrix $\mathcal{L}$ performs a boost along the x axis and leaves the other two dimensions unchanged. For our notes we will stick to the convention that greek indices label the components of four-vectors, while roman indices denote the components of three-vectors, i.e., $\alpha = 0, 1, 2, 3$ and $i = 1, 2, 3$.. The choice of making the indices upper vs. lower will be explained later.

For a boost along the x-axis, the 4×4 matrix becomes

$$\mathcal{L} = \left(\begin{array}{cccc} \gamma & \gamma v & 0 & 0 \\ \gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right). \tag{1.17}$$

For rotations of a coordinate system, all three vectors are transformed according to the same rotation matrix U, $r'_i = U_{ij} r_j$, regardless of whether the vector $\vec{r}$ refers to a spatial coordinate or a velocity or momentum. Similarly, for boosts all four-vectors are transformed the same,

$$r^\alpha = L^\alpha_{\ \beta} r^\beta, \tag{1.18}$$

regardless of whether the quantity r represents the space-time coordinate of an event or a momenta (in which case the zero-th component is the energy).

Due to the identity, $\gamma^2 - \gamma^2 v^2 = 1$, one can express γ and γv as $\cosh\eta$ and $\sinh\eta$ respectively. The Lorentz matrix then becomes

$$L = \left(\begin{array}{cccc} \cosh\eta & \sinh\eta & 0 & 0 \\ \sinh\eta & \cosh\eta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right), \tag{1.19}$$

which illustrates the similarity of the Lorentz transformations to rotations by an imaginary angle. Whereas for a rotation, one finds that $x_1y_1 + x_2y_2 + x_3y_3 = \vec{x} \cdot \vec{y}$ is invariant to rotations, Lorentz transformations leave the quantity,

$$r \cdot q \equiv r_0q_0 - r_1q_1 - r_2q_2 - r_3q_3 \tag{1.20}$$

unchanged. To see this consider the the boosted quantities,

$$\begin{aligned} r' &= (\gamma r_0 - \gamma v r_1, \gamma r_1 - \gamma v r_0, r_2, r_3), \\ q' &= (\gamma q_0 - \gamma v r q_1, \gamma r q_1 - \gamma v q_0, q_2, q_3). \end{aligned} \tag{1.21}$$

A little algebra shows that

$$\begin{aligned} r'_0q'_0 - r'_1q'_1 - r'_2q'_2 - r'_3q'_3 &= r_0q_0 - r_1q_1 - r_2q_2 - r_3q_3 \\ &= (\gamma^2 - \gamma^2 v^2)(r_0q_0 - r_1q_1) - r_2q_2 - r_3q_3 \\ &= r \cdot q. \end{aligned} \tag{1.22}$$

The "dot-product" of four vectors is as a Lorentz invariant, meaning a quantity unchanged by reference frame. The dot-product is also invariant to rotations in $3-$space, and the combinations of rotations and boosts is known as the *Lorentz group*. For a group, any combinations of transformations can be expressed as a single transformation. If one adds parity and time-reversal, it is known as the full Lorentz group, and if one adds translational symmetry, it becomes the Poincaré group. Rotations and boosts are not in separable groups, i.e., if one performs two boosts, the result is a boost plus a rotation. If the boosts had formed a group by themselves, any two boosts would have been equivalent to a single boost.

EXAMPLE:
Consider two successive boosts along the x direction. The first defined by $\gamma v = \sinh \eta_1$, and the second one with $\gamma v = \sinh \eta_2$. Show that the combination is equivalent to one boost with $\gamma v = \sinh(\eta_1 + \eta_2)$.

Writing down the product of the two Lorentz matrices (2-dimensions is sufficient),

$$\begin{aligned} &\left(\begin{array}{cc} \cosh\eta_2 & \sinh\eta_2 \\ \sinh\eta_2 & \cosh\eta_2 \end{array}\right)\left(\begin{array}{cc} \cosh\eta_1 & \sinh\eta_1 \\ \sinh\eta_1 & \cosh\eta_1 \end{array}\right) \\ &\qquad = \left(\begin{array}{cc} \cosh\eta_2\cosh\eta_1 + \sinh\eta_2\sinh\eta_1 & \cosh\eta_2\sinh\eta_1 + \sinh\eta_2\cosh\eta_1 \\ \cosh\eta_2\sinh\eta_1 + \sinh\eta_2\cosh\eta_1 & \cosh\eta_2\cosh\eta_1 + \sinh\eta_2\sinh\eta_1 \end{array}\right) \\ &\qquad = \left(\begin{array}{cc} \cosh(\eta_1+\eta_2) & \sinh(\eta_1+\eta_2) \\ \sinh(\eta_1+\eta_2) & \cosh(\eta_1+\eta_2) \end{array}\right), \end{aligned}$$

where double-angle formulas were applied for the last step. For high-energy phenomenology the the quantities η are referred to as "rapidities" when the boosts are along the beam axis. The simplicity of rapidities comes from the fact that they add like Newtonian velocities. However, this simple addition only works when the rapidities are defined along a single axis, i.e., if one defines $\sinh\eta_x = \gamma v_x, \cdots$, for all three dimensions, the addition formulas break down due to the non-commutation of boosts along different directions.

1.3 Invariants and the metric tensor $g^{\alpha\beta}$

As shown previously, the dot product of two vectors,

$$A^\alpha g_{\alpha\beta} B^\beta, \tag{1.23}$$

$$g_{\alpha\beta} \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{1.24}$$

is invariant to both rotations and boosts. As a notional trick, one can define four-vectors A_α (subscript rather than superscript) as

$$A_\alpha \equiv g_{\alpha\beta} A^\beta, \tag{1.25}$$

which means that $A^\alpha A_\alpha$ is an invariant (summation of repeated indices inferred). In all calculations the summed indices always appear with one superscript (contravariant) and one subscripted (covariant). The fact that multiplying by $g_{\alpha\beta}$ simply lowers the index also applies to tensors, i.e.,

$$g_{\alpha\beta} C^{\beta\gamma} = C_\alpha^{\ \gamma}. \tag{1.26}$$

Note that this property means that $g^\alpha_{\ \beta}$ is simply the unit matrix. Furthermore, this property also applies to $g^{\alpha\beta}$,

$$g^{\alpha\beta} C_{\beta\gamma} = C^\alpha_{\ \gamma}. \tag{1.27}$$

Derivatives might at first seem a little backwards. Consider a scalar function $\phi(x)$, where x is a four vector. For small changes in x, x

$$\delta\phi = \frac{\partial\phi}{\partial x^\mu} \delta x^\mu. \tag{1.28}$$

Since $\delta\phi$ is also a scalar, the vector $\partial/\partial x^\mu$ must transform as a covariant vector. This motivates the notation,

$$\partial_\mu = \frac{\partial}{\partial x^\mu}. \tag{1.29}$$

Note that for this convention, the equation of continuity is especially compact. For a four vector J, where J^0 refers to the charge density and J^i signals the current density, $\partial \cdot J = 0 \ \rightarrow \ \partial_t J_0 + \nabla \cdot \vec{J} = 0$. Maxwell equations also take on particularly beautiful forms, $\partial_\alpha F^{\alpha\beta} = J^\beta$, and $\partial_\alpha \tilde{F}^{\alpha\beta} = 0$. Additionally, we point out that we will employ the convention throughout this course that greek indices refer to all four components, while roman indices suggest only spatial components. Bold face will refer to the vector components of a three-vector, while four-vectors will not be put into bold face, i.e., $p \cdot x = p_0 x_0 - \mathbf{p} \cdot \mathbf{x}$.

1.4 Four-Velocities and Momenta

For a particle that moves between two points by a small displacement Δx^α, one can define the quantity,

$$\Delta\tau \equiv \sqrt{\Delta x_\alpha \Delta x^\alpha}, \tag{1.30}$$

which is an invariant. In the frame of the particle it is easy to see what this quantity represents, since in that frame the spatial components Δx_i are all zero. Thus $\Delta\tau$ is the amount a clock, moving with the particle, has progressed during the differential displacement. Further, one can define a vector,

$$u^\alpha \equiv \frac{\Delta x^\alpha}{\Delta\tau}, \tag{1.31}$$

which is also a four-vector since Δx^α is a four-vector and $\Delta\tau$ is a Lorentz scalar (or invariant). The four-vector u^α is often referred to as the relativistic velocity, and given that $\Delta x^0 = \gamma\Delta\tau$, one can see that

$$u^0 = \gamma,\ u^i = \gamma\frac{dx^i}{dt} = \gamma v^i. \tag{1.32}$$

It is easy to see that

$$u^\alpha u_\alpha = 1. \tag{1.33}$$

The momentum is defined by multiplying u^α by the particle's mass m, which is also a scalar. This then gives,

$$p^\alpha p_\alpha = m^2. \tag{1.34}$$

The zeroth component of the momentum is identified as the energy. This gives the relation that for a particle at rest,

$$E = p^0 = m. \tag{1.35}$$

Of course, the more famous relation, $E = mc^2$, requires keeping track of all the factors of c.

EXAMPLE:
A beam of particles of mass m_A is aimed a target with particles of mass m_B. What kinetic energy, K, is required so that one can make a resonance of mass m_C.

The solution is based on energy-momentum conservation. One way to move forward is to calculate the invariant mass in terms of the total momentum and set it to m_C.

$$\begin{aligned}
m_{\rm inv}^2 &= (p_A + p_B)^2 = (K + m_A + m_B)^2 - \vec{p}_A^{\,2}\\
&= (K + m_A + m_B)^2 - \left[(K + m_A)^2 - m_A^2\right]\\
&= m_A^2 + m_B^2 + 2m_B(K + m_A) = m_C^2,\\
K &= \frac{m_C^2 - (m_A + m_B)^2}{2m_B}.
\end{aligned}$$

EXAMPLE:
Consider two particles recorded with momenta p_A and p_B at space times separated by $r^\alpha = x_1^\alpha - x_2^\alpha$. In terms of relativistic invariants using p_A, p_B and r, solve for the impact parameter, i.e., the distance of closest approach as measured by an observer in the center-of-mass frame.

First, we consider two four vectors which will be used as projectors to eliminate the components of r that, in the center-of-mass frame, are either time-like or along the direction of the relative momentum. First, the time-like vector is the total momentum,

$$P^\alpha = p_A^\alpha + p_B^\alpha,$$

which in the c.o.m. frame becomes $(E_A + E_B, 0, 0, 0)$. One can define a vector,

$$r'^\alpha \equiv r^\alpha - P^\alpha \frac{P \cdot r}{P^2}.$$

In the center-of-mass frame, r' looks exactly like r, except the $\alpha = 0$ component vanishes. Next, one defines a vector which in the c.o.m. frame is parallel to the relative momentum, and is zero for the $\alpha = 0$ piece. This would be:

$$q'^\alpha = q^\alpha - P^\alpha \frac{P \cdot q}{P^2}, \;\; q^\alpha = p_A^\alpha - p_B^\alpha.$$

Again, one can consider the c.o.m. frame, where it is clear that $q'^0 = 0$ and $q'^i = (p_A^i - p_B^i)$. One can then project away the part of r' parallel to q, and make a new vector b,

$$b^\alpha \equiv r'^\alpha - q'^\alpha \frac{q' \cdot r'}{q'^2}$$

Finally, the impact parameter squared is, after some surprisingly painful algebra,

$$B^2 = -b^2 = -r^2 + \frac{(q \cdot r)^2 P^2 + (P \cdot r)^2 q^2 - 2(q \cdot r)(P \cdot r)(P \cdot q)}{P^2 q^2 - (P \cdot q)^2}.$$

1.5 Examples of Invariants

Here I discuss several invariants you will encounter at various times throughout this course or in the literature. First, one can see that d^4x is invariant to boosts by considering the Jacobian of a Lorentz transformation along the x axis. In that case,

$$\mathcal{L} = \begin{pmatrix} \cosh\eta & \sinh\eta & 0 & 0 \\ \sinh\eta & \cosh\eta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{1.36}$$

The Jacobian is simply the determinant of the matrix, $J = \cosh^2 - \sinh^2 = 1$. Thus, d^4x is invariant. For a boost in an arbitrary direction, one can first to a rotation (d^3x is invariant to rotation) combined with a boost.

Then d^4p is also invariant as is,

$$\int d^4p\, \delta(p^2 - m^2) s(p) = \int dp_0 d^3p\, \delta(p_0^2 - E_p^2) s(p) = \int \frac{d^3p}{2E_p}\; s(p)|_{p_0 = E_p}. \tag{1.37}$$

Here $s(p)$ is some arbitrary scalar function. Since s could be anything, then one can state that d^3p/E_p is also invariant. This is why spectra in particle physics are expressed as one of the two identical forms below

$$\frac{E_p dN}{d^3p} = \frac{dN}{d\phi p_t dp_t dy}. \tag{1.38}$$

The expression on the right-hand side can be found by seeing that $dp_z/E = dy$, where y is the rapidity. Since dN is the number of counts in a bin, and since a number of counts is invariant, you needn't worry about how dN transforms.

Often, when calculating the number of particles in a box, one writes

$$N = \int \frac{d^3x d^3p}{(2\pi)^3} f(p), \tag{1.39}$$

where f is the phase space density, or occupation probability. This must be an invariant. For instance, if you had fermions as zero temperature, f would be zero or unity depending on whether it was inside or outside the Fermi sea. This would not change if viewed from a different reference frame. Since f is invariant, and dN is invariant, it stands that $d^3x\ d^3p$ is also invariant. To see that this is true, one must remember that for this expression one assumes that dx measures the distance between the boundaries of a cell where both the positions of the boundaries are measured simultaneously, like measuring the length of a meter stick. If one compares the volume measured by an observer moving with the velocity of the particle, p/E, and compares to the length in the lab frame one finds that d^3x shorter by the Lorentz factor, E/m. If one considers d^3p in the two frames, one sees that d^3p in the lab frame must be larger by a factor of E/m so that d^3p/E is invariant. Thus, the product d^3pd^3x is invariant.

Another example that comes up often is a collision rate per volume and per time, dN_c/d^4x, which is manifestly invariant. One can also consider the collision rate of particles from specific regions of momentum space, d^3p_a/E_a and d^3p/E_b. The quantity would normally written as

$$E_aE_b\frac{dN_c}{d^4x\ d^3p_a\ d^3p_b} = f_a(p_a)f_b(p_b)Something, \tag{1.40}$$

where f_a and f_b are the phase space densities, or occupancies, of the two particles. Occupancies are also invariant, i.e. if the Fermi sea is full, $f = 1$ regardless of what frame is considered. Here, $Something$ has to be an invariant. Since it is invariant one can consider the center-of-mass frame where the momenta are back to back, $p_a = -p_b$. In that frame one knows from kinematics that

$$Something = \frac{E_aE_b}{(2\pi)^6}\sigma(\sqrt{s})v_{\rm rel}, \tag{1.41}$$

where σ is the cross section and the relative velocity is $v_{\rm rel} = |p|/E_a + |p|/E_b$. One must then simply write $E_aE_bv_{\rm rel}$ as a Lorentz invariant. To do this,

$$E_aE_bv_{\rm rel} = \vec{p}_aE_b - \vec{p}_bE_a|. \tag{1.42}$$

One can now write $E_a = p_a \cdot P/\sqrt{s}$ and replace p_a with the spatial components of $p'_a \equiv p_a - P(p_a \cdot P)/s$ to project out the zero$^{\rm th}$ components of p_a. Doing the same for p_b one then finds

$$\begin{aligned}
(E_aE_bv_{\rm rel})^2 &= -\left[(p_b\cdot P)(p_a-P(p_a\cdot P)/s)-(p_a\cdot P)(p_b-P(p_b\cdot P)/s)\right]^2/s \\
&\quad \cdots \text{LOTS of algebra}\cdots \qquad (1.43)\\
&= \frac{(s-m_a^2+m_b^2)(s+m_a^2-m_b^2)}{4s}\left[s^2+m_a^4+m_b^4-2m_a^2m_b^2-2sm_a^2-2sm_b^2\right]\\
&= \frac{(s-m_a^2+m_b^2)(s+m_a^2-m_b^2)}{s}\left[(p_a\cdot p_b)^2-m_a^2m_b^2\right],\\
&= wx
\end{aligned}$$

Here, numerous steps are omitted from the last step as this is related to a homework problem at the beginning of the last chapter. Putting this all together, one finds

$$\frac{dN_c}{d^4x} = \int \frac{d^3p_a}{(2\pi)^3E_a}\frac{d^3p_b}{(2\pi)^3E_b}\, f_a(p_a)f_b(p_b)\sigma(\sqrt{s})\sqrt{(p_a\cdot p_b)^2-m_a^2m_b^2}. \tag{1.44}$$

1.6 Tensors

Tensors are quantities with two or more Lorentz indices. Examples are the stress-energy tensor and the electromagnetic field tensor. Similarly to how rotations in three-dimensional space affect three-dimensional tensors,

$$M'_{im} = U_{ij}M_{jk}U^{-1}U_{km}, \tag{1.45}$$

a relativistic tensor translates as

$$M'^{\alpha\delta} = L^{\alpha\beta}M_{\beta\gamma}L^{(-1)\gamma\delta}. \tag{1.46}$$

Again, note that summed indices always involve one covariant and one contravariant index. So the above equation can be rewritten a number of ways, e.g.

$$M'^{\alpha}_{\ \delta} = L^{\alpha\beta}M_{\beta}^{\ \gamma}L(-1)_{\gamma\delta}. \tag{1.47}$$

An example of a fourth-rank tensor is the anti-symmetric tensor,

$$\epsilon_{\alpha\beta\gamma\delta} = \begin{cases} 1, & \text{for even permutations of } \alpha\beta\gamma\delta \text{ from } 0123,\\ -1, & \text{for odd permutations of } \alpha\beta\gamma\delta \text{ from } 0123,\\ 0, & \text{if any index repeats.}\end{cases} \tag{1.48}$$

With these definitions,

$$\begin{aligned}
\epsilon_{0123} &= \epsilon_{1230}=\epsilon_{2301}=\epsilon_{3012}=\epsilon_{0231}=\epsilon_{2310} \qquad (1.49)\\
&= \epsilon_{3102}=\epsilon_{1023}=\epsilon_{0312}=\epsilon_{3120}=\epsilon_{1203}=\epsilon_{2031}=1,\\
\epsilon_{1023} &= \epsilon_{0231}=\epsilon_{2310}=\epsilon_{2310}=\epsilon_{0132}=\epsilon_{1320}\\
&= \epsilon_{3201}=\epsilon_{2013}=\epsilon_{0213}=\epsilon_{2130}=\epsilon_{1302}=\epsilon_{3021}=-1,\\
\epsilon_{\alpha\beta\gamma\delta} &= 0 \text{ otherwise.}
\end{aligned}$$

1.7 Homework Problems

1. Suppose you are doing a fixed-target experiment at the LHC. The protons have a beam kinetic energy of 7 TeV. (The proton mass is 938.28 MeV/c^2). If the experiment were redone with a collider built with an equivalent center-of-mass energy, what would the kinetic energy of each beam be?

2. Find the equivalent fixed beam energy for a fixed target to have the same center-of-mass energy as the collider experiment at the LHC.

3. Consider a 1+1 dimension vector, (E, p), where $m^2 \equiv E^2 - p^2$. Consider the transformed vector, $p'_\alpha = L_{\alpha\beta} p^\beta$, where L is defined according to Eq. (1.19). Show that $m'^2 \equiv E'^2 - p'^2 = m^2$.

4. Consider two particles with four momenta p_a and p_b. Particle a is recorded at the space time point $r_a = (0, 0, 0, 0)$ and particle b is recorded at $r_b = r$. For an observer moving with particle a find the time at which particle b passes at the point of closest approach. Express your answer in terms of Lorentz invariants, i.e., dot products involving p_a, p_b and r.

5. Consider two particles of mass m_a and m_b with four momenta p_a and p_b. Show that the relative velocity, $|\vec{v}_a - \vec{v}_b|$, according to an observer in the center-of-mass frame is:

$$v_{\rm rel}^2 = \frac{s^2/4}{(p_a \cdot P)^2 (p_b \cdot P)^2} \left[s^2 + m_a^4 + m_b^4 - 2sm_a^2 - 2sm_b^2 - 2m_a^2 m_b^2\right].$$

   Here, P is the total momentum and $s = P^2$. FYI: The answer is sometimes called the triangle function. If you have a triangle with sides of length $(\sqrt{s}, m_a, m_b)$ and solve for the area of the triangle you get a similar function aside from the prefactor. This term for the relative velocity appears often, e.g. you divide by this factor to convert rates from Feynman diagrams into cross sections.

6. The Lorentz transformation is a tensor, $\mathcal{L}^{\alpha\beta}$, which transforms some four vector p^α observed by an observer moving with four velocity u^α to a vector p'^α as determined by an observer moving with four-velocity u'^α.

$$\mathcal{L}^\alpha_{\ \beta} p^\beta = p'^\alpha.$$

   Since $\mathcal{L}$ is a tensor it must be of the form,

$$\mathcal{L}^{\alpha\beta} = Au^\alpha u'^\beta + Bu'^\alpha u^\beta + Cu^\alpha u^\beta + Du'^\alpha u'^\beta + Eg^{\alpha\beta}, \qquad (1.50)$$

   where $A - E$ are scalar functions of u and u'. Since $u^2 = u'^2 = 1$, the only scalar function available is $u \cdot u'$. Consider the transformation from the rest frame $u = (1, 0, 0, 0)$ to the frame $u' = (\gamma, \gamma v, 0, 0)$. You know that the Lorentz transformation is:

$$\mathcal{L}^\alpha_{\ \beta} = \begin{pmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Further, $\boldsymbol{u} \cdot \boldsymbol{u}' = \gamma$. From the form in Eq. (1.50),

$$\boldsymbol{\mathcal{L}}^{\alpha}_{\ \beta} = \begin{pmatrix} A\gamma + B\gamma + C + D\gamma^2 + E & -A\gamma v - D\gamma^2 v & 0 & 0 \\ B\gamma v + D\gamma^2 v & -D\gamma^2 v^2 + E & 0 & 0 \\ 0 & 0 & E & 0 \\ 0 & 0 & 0 & E \end{pmatrix}.$$

(a) Solve for the coefficients $\boldsymbol{A}$ through $\boldsymbol{E}$ in terms of $\gamma = \boldsymbol{u} \cdot \boldsymbol{u}'$.

(b) Show that for the four vector $\boldsymbol{u}'$,

$$\boldsymbol{\mathcal{L}}^{\alpha\beta} \boldsymbol{u}'_{\beta} = \boldsymbol{u}^{\alpha}.$$

(c) Show that

$$\boldsymbol{\mathcal{L}}^{\alpha\beta}(\boldsymbol{u}, \boldsymbol{u}') \boldsymbol{\mathcal{L}}_{\beta\gamma}(\boldsymbol{u}', \boldsymbol{u}) = \boldsymbol{g}^{\alpha}_{\ \gamma}.$$

2 Dynamics of Relativistic Point Particles

We won't discuss the dynamics of individual charged particles very much in this course, but it is good to review the least-action principle, and the example for relativistic particles. In the next chapter we will apply these principles to field equations, deriving Maxwell's equations, so it will be helpful to review the connection between least action and Lagrange's equations, with an emphasis on how symmetries lead to conservation laws. In this chapter we consider the interaction with an external electromagnetic field. This chapter closely follows the approach in Landau and Lifshitz.

2.1 Lagrangian for a Free Relativistic Particle

The action, S, is related to a Lagrangian by

$$S = \int_{t_a}^{t_b} dt\ \mathcal{L}(\vec{r}(t), \dot{\vec{r}}(t), t). \tag{2.1}$$

Usually, there $\mathcal{L}$ has no explicit dependence on t and only depends on $\vec{r}$ and $\dot{\vec{r}}$. In that case minimizing S leads to Lagrange's equations,

$$\begin{aligned} \delta S &= \int_{t_a}^{t_b} dt\ \sum_i \left(\frac{\partial \mathcal{L}}{\partial \dot{r}_i} \delta \dot{r}_i + \frac{\partial \mathcal{L}}{\partial r_i} \delta r_i \right) \\ &= \int_{t_a}^{t_b} dt\ \sum_i \left(-\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}_i} + \frac{\partial \mathcal{L}}{\partial r_i} \right) \delta r_i \\ &= 0. \end{aligned} \tag{2.2}$$

The equivalence must be true for any δr_i at any time, thus one derives the usual Lagrange equations of motion,

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}_i} \delta r_i = \frac{\partial \mathcal{L}}{\partial r_i}. \tag{2.3}$$

If there are symmetries, in that $\mathcal{L}$ does not depend on some coordinate, a conservation law ensues. For instance, if $\mathcal{L}$ is independent of ϕ (but not $\dot{\phi}$) the rotational symmetry gives,

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = 0, \tag{2.4}$$

and $p_\phi = \partial\mathcal{L}/\partial\dot{\phi}$ is conserved. More generally, consider any small change $x_i \rightarrow x + \epsilon_i(x,t)$. For translational invariance ϵ is independent of x and $\delta x_i = \epsilon_i(t)$, and setting $\delta S = 0$, one finds

$$\begin{aligned} \delta S &= 0 \\ &= -\int_{t_a}^{t_b} dt\ \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}_i} \epsilon_i(t). \end{aligned} \tag{2.5}$$

Thus $\partial\mathcal{L}/\dot{x}_i$ is conserved, and is the momentum in the i direction. For rotational invariance about an axis $\hat{\Omega}$, $\delta x_i = \epsilon_{ijk}x_j\hat{\Omega}_k\delta\phi(t)$,

$$\begin{aligned} \delta S &= 0 \\ &= -\int_{t_a}^{t_b} dt\, \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{x}_i}\epsilon_{ijk}(t)x_j(t)\hat{\Omega}_k\delta\phi(t). \end{aligned} \tag{2.6}$$

This must be true for any small angle $\delta\phi$, and the conserved quantity is known as the angular momentum vector about the $\hat{\Omega}$ axis.

$$\begin{aligned} L_\Omega &= -\frac{\partial\mathcal{L}}{\partial\dot{x}_i}\epsilon_{ijk}x_j(t)\hat{\Omega}_k \\ &= (\vec{r}\times\vec{p})\cdot\hat{\Omega}. \end{aligned} \tag{2.7}$$

If there exists rotational invariance about any axis then all three components of $\vec{L}$ are conserved.

The connection between symmetries and conservation laws is as important a concept as any in all of physics. For example, in quantum field theory, the arbitrary phase of a complex field operator is related to conservation of electric charge.

Conservation of energy comes from Lagrange's equations. Defining $\pi_i \equiv \partial\mathcal{L}/\partial\dot{q}_i$, where q_i are the generalized coordinates,

$$\begin{aligned} \frac{d}{dt}\left(\pi_i\dot{q}_i - \mathcal{L}\right) &= \dot{\pi}_i\dot{q}_i + \pi_i\ddot{q}_i - \frac{\partial\mathcal{L}}{\partial\dot{q}}\ddot{q} - \frac{\partial\mathcal{L}}{\partial q_i}\dot{q}_i \\ &= 0. \end{aligned} \tag{2.8}$$

Thus, the Hamiltonian $H = \pi_i\dot{q}_i - \mathcal{L}$ is conserved. Note this was contingent on the lack of explicit time dependence in $\mathcal{L}$. Thus, invariance under translation in time is associated with energy conservation.

Now, we turn back to the problem at hand, the relativistic motion of a free-streaming particle. The action is a Lorentz-invariant, which greatly constrains what form it can have. For free particles it can only depend on velocity so a good guess for the form is

$$S = -m\int_{t_a}^{t_b}\sqrt{dt^2 - (d\vec{r})^2} = -m\int_{t_a}^{t_b} dt\,\sqrt{1-\left(\frac{d\vec{r}}{dt}\right)^2}. \tag{2.9}$$

The choice of the mass, $-m$, for the preceding factor is motivated by having the non-relativistic expansion have a term $mv^2/2$. The equations of motion become

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{r}_i} = \frac{m}{\sqrt{1-v^2}}\dot{r}_i \tag{2.10}$$

$$= \frac{\partial\mathcal{L}}{\partial r_i} = 0. \tag{2.11}$$

So the conserved momentum is the usual $m\gamma\vec{v}$. One can also calculate the energy,

$$\begin{aligned} H &= \pi_i\dot{q}_i - \mathcal{L} = m\gamma v^2 - \frac{m}{\gamma} \\ &= m\gamma(v^2 + (1-v^2)) = m\gamma. \end{aligned} \tag{2.12}$$

Of course, if you included the factors of c we would get $E = mc^2$ for $v = 0$.

Boost symmetries are a bit more difficult to consider due to the mixing of time and position. The best way to consider these is to define $t = r_0$ and $\vec{r}$ in one frame, then consider the boost of small velocities or rotations

$$r'^{\alpha} = r^{\alpha} + \delta\Omega^{\alpha\beta}(t) r_{\beta}. \tag{2.13}$$

Here, the tensor $\Omega^{\alpha\beta}$ is any anti-symmetric tensor. The three $0i$ elements represent boosts, while the three non-zero ij elements represent rotations.

Now, we rewrite the Lagrangian and action as

$$S = -m \int_{t_a}^{t_b} dt \sqrt{\left(\frac{dt'}{dt}\right)^2 - \left(\frac{d\vec{r'}}{dt}\right)^2}. \tag{2.14}$$

For $\delta v = 0$ the coordinates are $t' = t, r'_i = r_i$, which gives the usual answer. For small $\delta\vec{v}$,

$$\begin{aligned}
\delta S &= \int_{t_a}^{t_b} dt \frac{\partial \mathcal{L}}{\partial \dot{r}^{\alpha}} \frac{d}{dt}\left(\delta\Omega^{\alpha\beta} r_{\beta}\right) \\
&= \int_{t_a}^{t_b} dt \left[-\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}^{\alpha}} r_{\beta}\right) + \left(\frac{\partial \mathcal{L}}{\partial \dot{r}^{\alpha}}\right)\dot{r}_{\beta}\right]\delta\Omega^{\alpha\beta} \\
&= \int_{t_a}^{t_b} dt \left[-\frac{d}{dt}\left(\pi_{\alpha} r_{\beta}\right) + \pi_{\alpha}\dot{r}_{\beta}\right]\delta\Omega^{\alpha\beta} \\
&= 0.
\end{aligned} \tag{2.15}$$

The second term disappears because $\delta\Omega^{\alpha\beta}$ is anti-symmetric and π is parallel to $\dot{r}$. This must be zero for any contribution from given components $\delta\Omega_{\alpha\beta}$ and $\delta\Omega_{\beta\alpha} = -\delta\Omega_{\alpha\beta}$,

$$\frac{d}{dt}\left(r^{\alpha}p^{\beta} - p^{\alpha}r^{\beta}\right) = 0, \tag{2.16}$$

for any choice of $\alpha\beta$. For $\alpha\beta$ both with space-like indices, these components correspond to angular momenta.

$$\begin{aligned}
M^{\alpha\beta} &= r^{\alpha}p^{\beta} - p^{\alpha}r^{\beta}, \\
M^{12} &= L_z,\ \ M^{23} = L_x,\ \ M^{31} = L_y.
\end{aligned} \tag{2.17}$$

For the components with $\alpha = 0$,

$$M^{01} = tp_x - xE,\ M^{02} = tp_y - yE,\ M^{03} = tp_z - zE. \tag{2.18}$$

Each of the six elements represent conserved quantities. For the M^{0i} elements the quantities represent the fact that the center-of-mass moves with fixed velocity. If there were several particles the quantities would be

$$\sum_a (\vec{p}_a t - \vec{r}E_a) = \text{constant}. \tag{2.19}$$

Dividing the equation by the conserved energy, $E_{\rm tot} = \sum_a E_a$, one gets

$$\left(\frac{\sum_a \vec{p}_a}{E_{\rm tot}}\right) t \quad - \quad \frac{\sum_a E_a \vec{r}_a}{E_{\rm tot}} = \text{constant}. \tag{2.20}$$

Thus the center of mass is defined as an average sum over the positions weighted by their energies, and it moves with constant velocity $\vec{P}_{\rm tot}/E_{\rm tot}$. Of course, all this is postulated on the Lagrangian not having interactions that might destroy the Lorentz invariance assumed above.

2.2 Interaction of a Charged Particle with an External Electromagnetic Field

Here, the external electromagnetic field is a four-vector $A^\mu(r)$. The zeroth component is the electric potential ϕ and the spatial components are the usual vector potential. To make a Lorentz-invariant action that has a contribution that looks like the usual potential energy, $e\phi(r)$, we consider

$$\begin{aligned} S &= \int d\tau\, (-m - e u \cdot A) \qquad (2.21) \\ &= -m \int dt \sqrt{\left(\frac{dt'}{dt}\right)^2 - \left(\frac{d\vec{r}'}{dt}\right)^2} - e \int dt\, \left(A_0 - \vec{v}\cdot\vec{A}\right). \end{aligned}$$

Here, we have used the fact that $u_0 d\tau = \gamma d\tau = dt$ and $\vec{u} d\tau = \vec{v} dt$. The conjugate momenta is changed by the appearance of the velocity in the $\vec{v}\cdot\vec{A}$ term,

$$\vec{\pi} = m\gamma\vec{v} + e\vec{A}. \tag{2.22}$$

Lagrange's equations then become

$$\begin{aligned} \frac{d}{dt}(m\gamma v_i + eA_i) &= -e\partial_i A_0 + e\vec{v}\partial_i \cdot \vec{A} \qquad (2.23) \\ \frac{d}{dt} m\gamma v_i &= -e\partial_t A_i - e\partial_i A_0 - e\vec{v}\cdot\nabla A_i + e\vec{v}\partial_i \cdot \vec{A}. \end{aligned}$$

Here, we have made use of the fact that $d/dt = \partial_t + \vec{v}\cdot\nabla$. Next, the last two terms can be manipulated with the vector identity,

$$\begin{aligned} \vec{A}\times(\vec{B}\times\vec{C}) &= \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B}), \qquad (2.24) \\ \frac{d}{dt}(m\gamma v_i) &= -e\partial_i A_0 - e\partial_t A_i + e\vec{v}\times(\nabla\times\vec{A}). \end{aligned}$$

Thus, the electric and magnetic fields are

$$\begin{aligned} \vec{E} &= -\nabla A_0 - \partial_t \vec{A}, \qquad (2.25) \\ \vec{B} &= \nabla\times\vec{A}, \end{aligned}$$

and the equations of motion are

$$\frac{d}{dt}(m\gamma v_i) = e\vec{E} + e\vec{v}\times\vec{B}. \tag{2.26}$$

The electric potential is the zero$^{\rm th}$ component of the four-vector potential. It is **NOT** a scalar. Since the four components of A mix during a boost, magnetic and electric fields also mix. Consider a frame where there is a constant electric field in the $\hat{z}$ direction due to a potential,

$$A^0 = -zE, \quad \vec{A} = 0. \tag{2.27}$$

If one boosts in the x direction by a velocity v,

$$\begin{aligned} A'^0 &= \gamma(-zE), \\ A'^x &= \gamma v(-zE). \end{aligned} \tag{2.28}$$

Using the fact that $z = z'$ for a boost along the x axis, the ensuing electric and magnetic fields are

$$\begin{aligned} \vec{E}' &= \gamma E\hat{z}, \\ \vec{B}' &= \nabla \times \vec{A}' = -vE\hat{y}. \end{aligned} \tag{2.29}$$

2.3 Motion in a Constant Magnetic Field

Let's consider the gauge where

$$\vec{A} = xB\hat{y}, \tag{2.30}$$

which gives $\vec{B} = B\hat{z}$. Beginning with the action, we solve Lagrange's equations

$$S = -\int dt\, \left\{ m\sqrt{1-v^2} - exBv_y \right\}, \tag{2.31}$$

$$\begin{aligned} p_y &= m\gamma v_y + eBx, \quad = \text{constant because there is no } y \text{ dependence}, \\ p_x &= m\gamma v_x, \\ \frac{d}{dt}p_x &= eBv_y. \end{aligned} \tag{2.32}$$

Now, let's show that is indeed circular motion. We need to show that these equations (clockwise circular motion) are satisfied by

$$\begin{aligned} x &= x_0 + R\cos(\omega t - \phi), \\ y &= y_0 - R\sin(\omega t - \phi), \\ v_x &= -\omega R\sin(\omega t - \phi), \\ v_y &= -\omega R\cos(\omega t - \phi). \end{aligned} \tag{2.33}$$

From the equation for dp_x/dt one sees that

$$\omega = \frac{eB}{m\gamma}, \tag{2.34}$$

which is the usual expression for the cyclotron frequency except for the $1/\gamma$ factor. Note this is why cyclotrons can't work for relativistic energies.

Next, we must show that p_y is indeed a constant.

$$\begin{aligned} p_y &= -m\gamma V\cos(\omega t-\phi)+eBx, \\ &= -m\gamma\omega R\cos(\omega t-\phi)+eB(x_0+R\cos(\omega t-\phi)) \\ &= eBx_0.\ \checkmark \end{aligned} \tag{2.35}$$

This emphasizes that $\vec{p}$ in not the velocity multiplied by the mass, but differs by $e\vec{A}$. So having p_y being a constant does not mean that the y-component of the velocity is fixed.

2.4 Gauge Transformations

Consider the following change to the vector potential,

$$A'^{\mu} = A^{\mu}+\partial^{\mu}\Lambda(t,x,y,z). \tag{2.36}$$

Here Λ is a arbitrary scalar function. The electric and magnetic fields become (with the use of Eq. (2.40))

$$E_i' = E_i-\partial_i\partial_t\Lambda+\partial_t\partial_i\Lambda=E_i, \tag{2.37}$$
$$B_i' = B_i-\epsilon_{ijk}\partial_j\partial_k\Lambda=B_i. \tag{2.38}$$

Thus, E and B are unchanged by the function Λ, even though $\vec{A}$ is changed. This is known as gauge invariance. In nature, the fields $\vec{A}$ are indeed physical when one considers quantum mechanics, e.g. the Aharonov-Bohm effect. However, even then gauge invariance is still true the scalar function Λ is arbitrary.

2.5 The Electromagnetic Field Tensor

One can define a second-rank anti-symmetric tensor using the vector potential,

$$F^{\alpha\beta} = \partial^{\alpha}A^{\beta}-\partial^{\beta}A^{\alpha}. \tag{2.39}$$

Using the definitions,

$$\begin{aligned} \vec{E} &= -\partial_t\vec{A}-\nabla A_0, \\ \vec{B} &= \nabla\times\vec{A}, \end{aligned} \tag{2.40}$$

one can consider $F^{\alpha\beta}$ component-by-component and find

$$F_{\alpha\beta} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix},\quad F^{\alpha\beta}=\begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}. \tag{2.41}$$

The equations of motion then take the form (see HW problem),

$$m\frac{d}{d\tau}u^{\alpha} = eF^{\alpha\beta}u_{\beta}, \tag{2.42}$$

where $d\tau=\sqrt{dt^2-(d\vec{r})^2}$, is the differential time step in the frame of the particle, i.e. its proper time.

2.6 Homework Problems

1. Consider a system of free particles a with the Lagrangian set as

$$\mathcal{L} = -m_a \sum_a \sqrt{\left(\frac{dt'_a}{dt}\right)^2 - \left(\frac{d\vec{r}'_a}{dt}\right)^2}, \tag{2.43}$$

   with $t'_a = t$ before transforming. Now consider a translation in time,

$$t'_a = t + \epsilon(t), \quad \vec{r}'_a = \vec{r}_a.$$

   Calculate δS and express it so that δS is proportional to $\epsilon(t)$, not $\dot{\epsilon}$. Show that this quickly gives energy conservation.

2. Consider a particle of charge e and mass m moving in a constant electric field in the x direction, $A^0 = -eEx$. The particle's initial momentum is $\vec{p}(t=0) = p_y\hat{y}$.

   (a) Solving Lagrange's equation, find $p_x(t)$.

   (b) Using the fact that $v_x = p_x/\sqrt{m^2 + p_x^2 + p_y^2}$, find $x(t)$.

   (c) Using the fact that $v_y = p_y/\sqrt{m^2 + p_x^2 + p_y^2}$, find $y(t)$.

   (d) Find the trajectory, $x(y)$.

   (e) Take the limit $p_y/m << 1$ and find $x(y)$ again. Solve non-relativistically and compare.

3. Consider a particle of charge e and mass m moving in a constant magnetic field in the z direction. We will consider the gauge where,

$$A_y = xB/2, \tag{2.44}$$
$$A_x = -yB/2, \tag{2.45}$$
$$\vec{A} = \frac{B\rho\hat{\phi}}{2}. \tag{2.46}$$

   Here, $\rho = \sqrt{x^2 + y^2}$ and $\hat{\phi} = \hat{y}\cos\phi - \hat{x}\sin\phi$ is one of the three unit vectors in the cylindrical coordinate basis, $\hat{z}$, $\hat{\rho}$, and $\hat{\phi}$.

   (a) Find the scalar function Λ that transforms the choice used in the earlier section, $\vec{A} = xB\hat{y}$ to the vector potential used here.

   (b) Find $p_\phi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}}$.

   (c) Setting $\dot{z} = 0$, consider motion in the $x - y$ plane. Beginning with $(d/dt)\pi_\phi = 0$, show that the solution with fixed radius ($\dot{\rho} = 0$) will work if the $\dot{\phi}$ is constant and is related to the cyclotron frequency as shown in the previous subsection,

$$\dot{\phi} = \frac{eB}{m\gamma}.$$

4. Consider a particle of mass m moving in a scalar field, $\Phi(\vec{r}) = -Fx$. This simply changes the local mass to $m = m_0 - Fx$, and the Lagrangian is

$$\mathcal{L} = -(m_0 - Fx)\sqrt{1 - v^2}.$$

The mass changes with time as $m = m_0 - Fx$, and $dm/dt = -vF$, where v will be the velocity in the x direction. For the questions below, assume there is no movement in the y or z directions.

(a) Using Lagrange's equations, show

$$\frac{dv}{dt} = (1 - v^2)\frac{F}{m(t)}.$$

(b) Consider a particle at rest at $x = y = z = t = 0$. Show that the solutions to the above equations are

$$\begin{aligned} m &= m_0 \cos(at), \quad a = F/m_0 \\ v &= \sin(at), \\ x &= \frac{1}{a}(1 - \cos(at)). \end{aligned}$$

(c) At what time, $t_{\max}$, does the mass become zero?

(d) What is $x(t_{\max})$?

(e) What is $v(t_{\max})$?

(f) What is $u_x = \gamma v$ at $t_{\max}$?

5. It is easy to derive the spatial components of Eq. (2.42),

$$\frac{d}{d\tau}p^i = qF^{i\alpha}u_\alpha,$$

by expressing $F^{\alpha\beta}$ in terms of $\vec{E}$ and $\vec{B}$ using Eq. (2.41), then comparing the Eq. (2.26). It is less obvious to see how one obtains the equation for $d/d\tau p^0$. Given that $p^0 = m\gamma$, one can quickly see that

$$\frac{d}{d\tau}p^0 = e\vec{v} \cdot \frac{d}{d\tau}\vec{p}.$$

Beginning with the two expressions above, show that

$$\frac{d}{d\tau}p^0 = qF^{0\beta}u_\beta.$$

Hint, you will need to use the fact that $F^{\alpha\beta}$ is anti-symmetric.

3 Dynamic Electromagnetic Fields

So far we have discussed the motion of particles in a field, but have ignored how the fields might change in time. To do so, we need to consider all three parts of the action: the action of a free particle, S_m, the action involving the interaction of matter with the field S_{fm} and, new for this chapter, the action of the field, S_f. We will show how this action leads to Maxwell's equations.

3.1 Lagrangian (Density) for Free Fields: Deriving Maxwell's Equations

In the last chapter we considered the motion of a particle in a field, $A^\alpha(r)$, where the field was given as a function of time. First we will derive Maxwell's equations, which describe how the field responds to the current and how it evolves. For the moment, we ignore the external currents and look at S_f,

$$S_f = \int d^4r \mathcal{L}(A^\mu, \partial_\nu A^\mu), \tag{3.1}$$

where A and $\partial_\nu A$ are functions of the four-position r. Here $\mathcal{L}$ is not actually the Lagrangian, it is the Lagrangian density, and $\int d^3r \mathcal{L}$ is the Lagrangian. However, people typically refer to it as the Lagrangian anyway. A difference between the Lagrangian for a point particle is that $\mathcal{L}$ is a function of all four derivatives, $\partial_t A^\mu, \partial_x A^\mu, \partial_y A^\mu$ and $\partial_z A^\mu$. Again, we start with the condition of minimizing the action,

$$\begin{aligned} \delta S &= \int d^4r \left\{ (\partial_\mu \delta A^\nu) \frac{\partial \mathcal{L}}{\partial \partial_\mu A^\nu} + (\delta A^\nu) \frac{\partial \mathcal{L}}{\partial A^\nu} \right\} \\ &= \int d^4r \, \delta A^\nu \left\{ -\partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu A^\nu} + \frac{\partial \mathcal{L}}{\partial A^\nu} \right\}. \end{aligned} \tag{3.2}$$

Minimizing the action gives Lagrange's field equations,

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu A^\nu} = \frac{\partial \mathcal{L}}{\partial A^\nu}. \tag{3.3}$$

Again, this assumes that $\mathcal{L}$ has no explicit dependence on r, as it only depends on r through its dependence on A and derivatives of A. The only visual difference between the usual Lagrangian equations and what we see here is that d/dt is replaced by ∂_μ.

Our next step is to write the Lagrangian density for free fields. The form must be Lorentz invariant, and must ultimately lead to the usual expressions for the energy density. Another criteria is that it is gauge-invariant, i.e. that it should depend only on $F^{\mu\nu}$, and not A. The choice that works is

$$\mathcal{L} = -\frac{1}{16\pi} F^{\mu\nu} F_{\mu\nu} = \frac{1}{16\pi} F^{\mu\nu} F_{\nu\mu}. \tag{3.4}$$

The last step used the fact that $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ is anti-symmetric. From Lagrange's field equations, Eq. (3.3),

$$\begin{aligned}
\partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu A^\nu} &= \frac{1}{16\pi}\partial_\mu \frac{\partial}{\partial \partial_\mu A^\nu}(\partial^\alpha A^\beta - \partial^\beta A^\alpha)(\partial_\alpha A_\beta - \partial_\beta A_\alpha) \\
&= \frac{1}{4\pi}\partial_\mu \left(F^{\mu\nu}\right) = 0.
\end{aligned} \tag{3.5}$$

If one adds the interaction between the fields and particles,

$$S_{fm} = \int d^4r \; J \cdot A, \tag{3.6}$$

Lagrange's field equations then become

$$\frac{1}{4\pi}\partial_\mu \left(F^{\mu\nu}\right) \;=\; J^\nu. \tag{3.7}$$

Here, the current J^α is the current density which is a four-vector. For charged particles moving in a small volume Ω,

$$J^\alpha = \frac{1}{\Omega}\sum_{a\in\Omega} q_a \frac{u_a^\alpha}{\gamma_a}. \tag{3.8}$$

For J^0 this $u_0/\gamma = 1$ and one sees that J^0 is the charge density. For the spatial components, $\vec{u}/\gamma = \vec{v}$ and one sees that J^i is the current density. One can also check to see that $J^\alpha(r)$ is a four-vector by writing it as

$$\begin{aligned}
J^\alpha(r) &= \sum_a \int d\tau_a \; \delta^4(r_a - r) q_a u_a^\alpha, \\
&= \sum_a \delta^3(\vec{r_a}(r_0) - \vec{r})\frac{q_a u_a^\alpha}{\gamma}.
\end{aligned} \tag{3.9}$$

If one averages this over some small volume Ω by integrating over the volume and dividing by Ω, one obtains Eq. (3.8). Thus, Eq.s (3.8) and (3.9) are equivalent expressions of the four-current.

Using Eq. (2.41) to express $F^{\mu\nu}$ in terms of $\vec{E}$ and $\vec{B}$, these four equations (one equation for each value of ν) then become

$$\begin{aligned}
\nabla \cdot \vec{E} &= 4\pi J^0, \\
(\nabla \times \vec{B}) - \partial_t \vec{E} &= 4\pi \vec{J}.
\end{aligned} \tag{3.10}$$

The zero$^{\text{th}}$ component of the current is the charge density, so this are simply a statement of two of Maxwell's equations. To obtain the other two equations we consider the tensor

$$\tilde{F}^{\mu\nu} \;=\; \frac{1}{2}e^{\mu\nu\alpha\beta}F_{\alpha\beta}, \tag{3.11}$$

with $\tilde{F}$ known as the *dual electromagnetic tensor*. One can them see that

$$\begin{aligned}
\partial_\mu \tilde{F}^{\mu\nu} &= -\frac{1}{2}\epsilon^{\nu\mu\alpha\beta}\partial_\mu(\partial_\alpha A_\beta - \partial_\beta A_\alpha) \\
&= 0.
\end{aligned} \tag{3.12}$$

Equating to zero comes from the fact that two derivatives, $\partial_\mu\partial_\alpha$ or $\partial_\mu\partial_\beta$, are contracted through the Levi-Civita tensor, which being anti-symmetric must cancel the contribution from the derivatives. Using Eq. (2.41) one can see that

$$\tilde{F}^{\alpha\beta} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & -E_z & E_y \\ B_y & E_z & 0 & -E_x \\ B_z & -E_y & E_x & 0 \end{pmatrix}. \tag{3.13}$$

Expressing Eq. (3.12) in terms of $\vec{E}$ and $\vec{B}$,

$$\begin{aligned}
\nabla \cdot \vec{B} &= 0, \\
\partial_t \vec{B} + \nabla \times \vec{E} &= 0.
\end{aligned} \tag{3.14}$$

Summarizing, Maxwell's equations in covariant notation are

$$\begin{aligned}
\partial_\alpha F^{\alpha\beta} &= 4\pi J^\beta, \\
\partial_\alpha \tilde{F}^{\alpha\beta} &= 0.
\end{aligned} \tag{3.15}$$

For a point charge within a sphere of radius R, Gauss' law says that

$$4\pi \int d^3r\, J^0 \quad + \quad \oint d\vec{A} \cdot \vec{E}. \tag{3.16}$$

Using symmetry, $\vec{E} = E\hat{r}$, and combined with the recognition that integrating the charge density inside the sphere gives the charge

$$Q = \int d^3r J^0, \tag{3.17}$$

one finds

$$\begin{aligned}
4\pi Q &= 4\pi R^2 E, \\
E &= \frac{Q}{R^2}.
\end{aligned} \tag{3.18}$$

Compared to the expressions of Coulomb's law to which one is usually accustomed, there is no $4\pi\epsilon_0$ in the denominator or constant k in the denominator. This is due to the way in which charge is defined. For example, if you read texts in nuclear physics you will typically find the Coulomb energy for a proton near a nucleus of atomic number Z written as

$$PE = \frac{Ze^2}{r}. \tag{3.19}$$

If one wishes the energy to be in MeV, and the radius r to be in Fermi (aka femtometers), one uses the fact that

$$e^2 = \frac{\hbar c}{137.036}, \tag{3.20}$$

where $e^2/\hbar c = 1/137.036$ is known as the fine structure constant, which is dimensionless, and $\hbar c = 197.327$ MeV fm. In other fields one might more typically see a more arbitrary definition of charge, e.g. Coulombs. In such cases, Coulomb's law must then be modified by some prefactor, e.g. $1/4\pi\epsilon_0$.

3.2 Pseudo-Vectors and Pseudo-Scalars

The vector potential A^α is a four-vector, and the field tensor $F^{\alpha\beta}$ is a second-rank tensor. The electric field $\vec{E} = -\nabla A_0 + \partial_t \vec{A}$ is a 3-vector, but one that transforms as part of a tensor if the transformation involves a boost. The magnetic field $\vec{B} = \nabla \times \vec{A}$ is a *pseudo-vector*. The "pseudo" comes from the fact that its definition, $B_i = \epsilon_{ijk}\partial_j A_k$, involves two-vectors, so even though it has only one vector index, it does not switch sign under parity $(x \rightarrow -x, y \rightarrow -y, z \rightarrow -z)$. Tensors can behave rather strangely under parity because some of the elements don't change under parity while others do. One can define the *pseudo-scalar*

$$F^{\alpha\beta}\tilde{F}_{\alpha\beta} = -4\vec{E}\cdot\vec{B}. \tag{3.21}$$

Like scalars, this is manifestly invariant under Lorentz transformation, however it is odd under parity. This can also be written as

$$\begin{aligned} F^{\alpha\beta}\tilde{F}_{\alpha\beta} &= \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}F_{\alpha\beta}F_{\gamma\delta} \\ &= \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}(\partial_\alpha A_\beta)(\partial_\gamma A_\delta). \end{aligned} \tag{3.22}$$

Each term has a product of four four-vector components, one being a zero$^{\rm th}$ component and the other three being spatial. Thus, this quantity is odd under parity.

Finally, another obvious example of a scalar, which is even under parity, is

$$F^{\alpha\beta}F_{\alpha\beta} = -2(|\vec{E}|^2 - |\vec{B}|^2). \tag{3.23}$$

Thus, although $\vec{E}$ and $\vec{B}$ mix under boosts, the difference of their magnitudes remains fixed.

The sign of a pseudo-vector or pseudo-scalar changes if one changes from a right-handed to a left-handed coordinate system. This is because ϵ_{ijk} was arbitrarily defined so that ϵ_{xyz} was positive. Even though magnetic forces feature pseudo-vectors, the interaction conserves parity, i.e. it does not matter whether you used a right-handed or left-handed coordinate system. This is because the force, which is something you can observe, behaves as $\nabla \times \vec{B}$, $F_i = q\epsilon_{ijk}v_j B_k$. Thus, in considering a force, the Levi-Civita symbol appears twice, once in defining $\vec{B}$ and once in defining the force. Thus, one would see the same effect using a left-handed coordinate system.

The weak interaction does not conserve parity. If one aligns a nucleus so that its angular momentum, which is a pseudo-vector, points along the positive $\hat{z}$, and if that nucleus undergoes a weak-interaction decay (beta decay), the electrons are emitted with strong preference parallel to the angular momentum. This violates parity because the direction of the electrons is real, it makes a difference if the electrons go up vs. down, but the direction of the angular momentum (or the magnetic field used to polarize the atoms) is a pseudo-vector and dependent on the right-handed vs. left-handed choice of coordinate system.

3.3 The Stress-Energy Tensor of the Electromagnetic Field

Just like the energy, $H = \pi\dot{q} - \mathcal{L}$, is conserved for a usual Lagrangian, one can define a similar quantity for the Lagrangian density. However, this quantity will be a second-rank tensor and will express local conservation of both energy and momentum. Just to reduce the index overload, we consider a function $\mathcal{L}(\phi, \partial_\alpha\phi)$. We can replace ϕ with A^μ, but essentially the proof will be the same aside from summing over each of the four fields A^μ. First, we define the stress-energy tensor $T^{\alpha\beta}$,

$$\begin{aligned} T^{\alpha\beta} &= \pi^\alpha\partial^\beta\phi - g^{\alpha\beta}\mathcal{L}, \\ \pi^\alpha &\equiv \frac{\partial\mathcal{L}}{\partial(\partial_\alpha\phi)} \end{aligned} \tag{3.24}$$

When $\alpha = \beta = 0$, this looks like the usual definition of energy, except this is the Lagrangian density so T^{00} has dimensions of energy per length cubed. To see how this is related to a conserved quantity, consider

$$\partial_\alpha T^{\alpha\beta} = (\partial_\alpha\pi^\alpha)(\partial^\beta\phi) + \pi^\alpha(\partial_\alpha\partial^\beta\phi) - \frac{\partial\mathcal{L}}{\partial\phi}\partial^\beta\phi - \frac{\partial\mathcal{L}}{\partial(\partial_\alpha\phi)}\partial_\alpha\partial^\beta\phi. \tag{3.25}$$

The first and third terms vanish via Lagrange's field equations, Eq. (3.3), and the second and fourth terms cancel via the definition of the conjugate momentum, π^α, in Eq. (3.24). Thus,

$$\partial_\alpha T^{\alpha\beta} = 0. \tag{3.26}$$

Any function J^μ that satisfies the equation of continuity $\partial_\mu J^\mu = 0$ implies that J is a conserved current density. For such a four-vector, the zero$^{\text{th}}$ component is the charge density and the three spatial components J^i are current densities. The conservation ensues because

$$\partial_t \int d^3r J_0 = -\int d^3r \nabla \cdot \vec{J}. \tag{3.27}$$

The latter terms vanishes because the currents vanish at infinity so the net charge $Q = \int d^3r J_0$ is conserved. The conservation is local because if one considers a small volume, the change in the charge equals the flux of current through the boundary, also seen via the divergence theorem (aka Gauss' law). The "charge" needn't be electric charge but any conserved quantity.

In our case, for each value of β in $T^{\alpha\beta}$ one has a conserved four-current. Thus, T^{00}, T^{01}, T^{02} and T^{03} represent densities of conserved quantities. Those quantities are the energy and momentum densities. The quantities T^{10}, T^{20} and T^{30} represent the flux of energy density, i.e.

$$\partial_t \oint d^3r\, T_{00} = -\oint d\vec{A} \cdot \vec{J}, \quad J_\alpha = T^{\alpha 0}. \tag{3.28}$$

The three quantities T^{0i} represent momentum densities. It is not obvious, but the stress-energy tensor is symmetric, and the flux of the energy, T^{i0} equals the momentum density, T^{0i}. To prove the symmetry once must consider the effect of an asymmetric term (further ahead). In this case the one finds an infinite angular acceleration, and angular momentum is not conserved.

The other nine components of the stress-energy tensor represent the flux of momentum. For example, the quantity $T^{xy}dS_y$ represents the rate momentum P_x flows through a surface element of area dS_y which points in the y direction. This would be a shear. In hydrodynamics, this vanishes and in the frame of the fluid (where $T^{0i} = 0$) $T^{ij} = P\delta_{ij}$, where P is the pressure. For field equations, the spatial components T^{ij} simply represent momentum fluxes and can be quite complicated, thought the tensor does have to remain symmetric.

Now, to return to the specific case of the electromagnetic field. In that case, the quantities π^α and $T^{\alpha\beta}$ discussed above are functions of four fields and to express the conjugate momenta or the stress-energy tensor, one must simply extend the above relations to sums over all four fields. There are four conjugate momenta for each field, and because A^α is effectively four fields, there are four conjugate momenta for each α and the momenta are represented by a four-by-four tensor.

$$\begin{aligned} \pi^{\alpha\beta} &= \frac{\partial \mathcal{L}}{\partial(\partial_\alpha A_\beta)}, \\ T^{\alpha\gamma} &= \pi^{\alpha\beta}\partial^\gamma A_\beta - g^{\alpha\gamma}\mathcal{L}. \end{aligned} \tag{3.29}$$

Using $\mathcal{L} = -\frac{1}{16\pi}F^{\alpha\beta}F_{\alpha\beta}$,

$$\begin{aligned} \pi^{\alpha\beta} &= \frac{-1}{4\pi}F^{\alpha\beta}, \\ T^{\alpha\gamma} &= \frac{-1}{4\pi}F^{\alpha\beta}\partial^\gamma A_\beta + \frac{1}{16\pi}g^{\alpha\gamma}F^{\mu\nu}F_{\mu\nu}. \end{aligned} \tag{3.30}$$

However, as previously stated the stress-energy tensor should be symmetric. To make it symmetric, we add a derivative of the form $\partial_\mu G^\mu$, which by Gauss's law will integrate to zero if one considers all space. Thus if we add the term, $(1/4\pi)\partial_\beta(F^{\alpha\beta}A_\gamma)$, the stress-energy tensor becomes

$$\begin{aligned} T^{\alpha\gamma} &= \frac{-1}{4\pi}F^{\alpha\beta}\partial^\gamma A_\beta + \frac{1}{16\pi}g^{\alpha\gamma}F^{\mu\nu}F_{\mu\nu} + \frac{1}{4\pi}\partial_\beta(F^{\alpha\beta}A^\gamma) \\ &= \frac{1}{4\pi}F^{\alpha\beta}F_\beta{}^\gamma + \frac{1}{16\pi}g^{\alpha\gamma}F^{\mu\nu}F_{\mu\nu} + \frac{1}{4\pi}(\partial_\beta F^{\alpha\beta})A^\gamma. \end{aligned} \tag{3.31}$$

The last term vanishes for free fields because $\partial_\alpha F^{\alpha\beta} = 0$ when there are no currents present, so

$$T^{\alpha\gamma} = \frac{1}{4\pi}F^{\alpha\beta}F_\beta{}^\gamma + \frac{1}{16\pi}g^{\alpha\gamma}F^{\mu\nu}F_{\mu\nu}. \tag{3.32}$$

In addition to being symmetric, the tensor is traceless, $T^\alpha{}_\alpha = 0$. Expressing the components in terms of $\vec{E}$ and $\vec{B}$,

$$\begin{aligned}
T^{00} &= \frac{1}{8\pi}(E^2+B^2), \qquad (3.33)\\
T^{0i} &= \frac{1}{4\pi}\epsilon_{ijk}E_jB_k,\\
T^{ij} = -T^i{}_j &= \frac{1}{8\pi}(\delta_{ij}(E^2+B^2) - 2E_iE_j - 2B_iB_j).
\end{aligned}$$

The momentum density, or energy flux is T^{0i} or writing the three components as a vector, $(\vec{E}\times\vec{B})/4\pi$. This is known as the Poynting vector.

To explain why the stress-energy tensor must be symmetric, consider an infinitesimal cube of dimension $a \times a \times a$. Consider rotation about the z axis. The shear forces, $T_{i\neq j}$, contribute to the angular momentum. The forces on the four sides due to T_{xy} and T_{yx} are $T_{yx}a^2\hat{x}$ on the upper face, $-T_{yx}a^2\hat{x}$ on the lower face, a$-T_{xy}a^2\hat{y}$ on the right face and $T_{xy}a^2\hat{y}$ on the left-side face. The net torque is thus

$$\tau = a^3(T_{yx} - T_{xy}). \qquad (3.34)$$

However, the moment of inertia scales as a^5, so the angular acceleration would scale as $1/a^2 \rightarrow \infty$, unless the tensor is anti-symmetric. To understand why $T_{0i} = T_{i0}$, one can consider a boost. Under boosts, any such asymmetry would translate into an asymmetry in the ij components.

One can also show explicitly that the stress-energy tensor is conserved in the presence of interactions with currents. Beginning with the definition for the field contribution, $T^{(\mathrm{f})}$, in Eq. (3.32),

$$\begin{aligned}
T^{(\mathrm{f})\alpha\beta} &= \frac{1}{4\pi}F^{\alpha\gamma}F_\gamma{}^\beta + \frac{1}{16\pi}g^{\alpha\beta}F^{\mu\nu}F_{\mu\nu}, \qquad (3.35)\\
\partial_\alpha T^{(\mathrm{f})\alpha\beta} &= \frac{1}{4\pi}(\partial_\alpha F^{\alpha\gamma})F_\gamma{}^\beta + \frac{1}{4\pi}F^{\alpha\gamma}\partial_\alpha F_\gamma{}^\beta + \frac{1}{8\pi}F^{\alpha\gamma}\partial^\beta F_{\alpha\gamma},\\
&= J^\gamma F_\gamma{}^\beta + \frac{F^{\alpha\gamma}}{4\pi}\left(\partial_\alpha\partial_\gamma A^\beta - \partial_\alpha\partial^\beta A_\gamma + \frac{1}{2}\partial^\beta\partial_\alpha A_\gamma - \frac{1}{2}\partial^\beta\partial_\gamma A_\alpha\right)\\
&= J^\gamma F_\gamma{}^\beta + \frac{F^{\alpha\gamma}}{4\pi}\left(\partial_\alpha\partial_\gamma A^\beta - \frac{1}{2}\partial^\beta\partial_\alpha A_\gamma - \frac{1}{2}\partial^\beta\partial_\gamma A_\alpha\right)
\end{aligned}$$

The terms inside the parenthesis vanish because they are explicitly symmetric in the $\alpha\gamma$ indices, which when contracted with the anti-symmetric tensor $F^{\alpha\gamma}$, must vanish. Thus,

$$\partial_\alpha T^{(\mathrm{f})\alpha\beta} = J_\gamma F^{\gamma\beta}. \qquad (3.36)$$

Next, we consider the contribution to the stress energy tensor from the matter contribution. To do this we consider matter to have a mass density

$$\mu(\vec{r}) = \sum_a m_a\delta(\vec{r}-\vec{r}_a), \qquad (3.37)$$

which given that the total mass is conserved, for free particles, one can write a conserved mass current,

$$\begin{aligned}
J^0 &= \mu(\vec{r}), \qquad (3.38)\\
\vec{J}^\alpha(r) &= \sum_a m_a\delta(\vec{r}-\vec{r}_a(r_0))u^\alpha(r)/u_0(r).
\end{aligned}$$

To see that J is indeed a four-vector, one could write

$$\begin{aligned} J^\alpha(r) &= \sum_a m_a \int d\tau_a \delta(r_0 - t_a)\delta(\vec{r} - \vec{r}_a(t_a)) u_a^\alpha \\ &= \sum_a m_a \delta(\vec{r} - \vec{r}_a(r_0)) u_a^\alpha / \gamma_a \\ &= \sum_a m_a \delta(\vec{r} - \vec{r}_a(r_0)) u^\alpha(r)/u_0(r). \end{aligned} \tag{3.39}$$

Here, the four-velocity dependence on $\vec{r}$ is some complicated function that has the correct velocity for each particle a when $r = r_a$. The mass current is conserved, $\partial \cdot J = 0$. The stress energy tensor can be written as

$$\begin{aligned} T^{(\mathrm{m})\alpha\beta}(r) &= J^\alpha(r) u^\beta(r), \\ \partial_\alpha T^{(\mathrm{m})\alpha\beta} &= (\partial \cdot J) u^\beta + (J \cdot \partial) u^\beta(r). \end{aligned} \tag{3.40}$$

The first term vanishes due to current conservation, and the second term becomes

$$\partial_\alpha T^{(\mathrm{m})\alpha\beta} = \sum_a m_a \delta(\vec{r}_a(t) - \vec{r})(1/u_a^0)(u_a \cdot \partial) u_a^\beta. \tag{3.41}$$

The derivative $u_a \cdot \partial$ is a Lorentz invariant, so it can be considered in the rest frame. In that frame it becomes $d/d\tau_a$, where τ_a is the proper time for the particle a. From the equations of motion for a particle of charge q_a in an electromagnetic field, Eq. (2.42),

$$\begin{aligned} \partial_\alpha T^{(\mathrm{m})\alpha\beta} &= \sum_a m_a \delta(\vec{r}_a(t) - \vec{r}) q_a F^{\beta\gamma} u_{a,\gamma}/(u_a^0 m_a) \\ &= \sum_a q_a \delta(\vec{r}_a(t) - \vec{r}) F^{\beta\gamma} u_{a,\gamma}/u_a^0 \\ &= F^{\beta\gamma} J_\gamma = -J_\gamma F^{\gamma\beta}. \end{aligned} \tag{3.42}$$

Thus, using Eq. (3.36), the sum of the field and matter contribution vanishes,

$$\partial_\alpha \left(T^{(\mathrm{f})\alpha\beta} + T^{(\mathrm{m})\alpha\beta} \right) = 0. \tag{3.43}$$

Conspicuous by its absence is the part of the action that represents the coupling between the current and the field, $-J \cdot A$. Indeed this would contribute a third portion of the stress-energy tensor. However, when we wrote down the field-part of the contribution there was a step where a term $J^\alpha A^\gamma$ was discarded due to being in a field-free region. If included, it would cancel the contributions to the energy density from the $J \cdot A$ term. The "field" energy effectively accounted for the $J \cdot A$ part of the Lagrangian by ignoring the last term in Eq. (3.32). The energy density, T_{00} is thus

$$T^{00} = \frac{1}{8\pi}(|\vec{E}|^2 + |\vec{B}|^2) + T^{(\mathrm{m})00}, \tag{3.44}$$

where the matter contribution is the kinetic energy of the particles. For a static charge distribution, one expected the energy to have terms of the form,

$$\Delta PE = \sum_{a<b} \frac{q_a q_b}{|\vec{r}_a - \vec{r}_b|}. \tag{3.45}$$

The mystery of the missing interaction comes from showing that the change of the electric field energy due to bringing the two charges from infinity to their current locations is precisely the expected value. To see this, we can consider the field energy of two charges q_a and q_b brought to within a relative distance R. The field energy is

$$U^{(\mathrm{f})} = \frac{1}{8\pi}\int d^3r\,(\vec{E}_a+\vec{E}_b)^2, \tag{3.46}$$

where $\vec{E}_{a,b}$ are the electric fields due to the two charges. We are interested to the change in $U^{(\mathrm{f})}$ due to moving the charges, and can thus worry only about the term from $\vec{E}_a\cdot\vec{E}_b$,

$$\begin{aligned}
\Delta U^{(\mathrm{f})} &= \frac{1}{4\pi}\int d^3r\,\vec{E}_a\cdot\vec{E}_b \qquad (3.47)\\
&= \frac{q_aq_b}{4\pi}\int d^3r\,\frac{\vec{r}\cdot(\vec{r}-\vec{R})}{r^3|\vec{r}-\vec{R}|^3}\\
&= \frac{q_aq_b}{4\pi}\int d^3r\,\nabla\frac{1}{r}\cdot\nabla\left(\frac{1}{|\vec{r}-\vec{R}|}\right)\\
&= -\frac{q_aq_b}{4\pi}\int d^3r\,\frac{1}{r}\,\nabla^2\left(\frac{1}{|\vec{r}-\vec{R}|}\right)\\
&= \frac{q_aq_b}{4\pi}\int d^3r\,\frac{1}{r}\,4\pi\delta(\vec{r}-\vec{R}) \qquad (3.48)\\
&= \frac{q_qq_b}{R}.
\end{aligned}$$

For an array of charges, the potential energy is a then the sum over the potential energy of each pair, as expected.

One can also quickly show the usual description of the energy being related to the charge density convoluted with the the electric potential,

$$\begin{aligned}
U^{(\mathrm{f})} &= \frac{1}{8\pi}\int d^3r\,\vec{E}\cdot\vec{E} \qquad (3.49)\\
&= \frac{1}{8\pi}\int d^3r(-\nabla A_0(\vec{r}))\cdot\vec{E}\\
&= \frac{1}{8\pi}\int d^3r A_0(\vec{r})\nabla\cdot\vec{E}\\
&= \frac{1}{2}\int d^3r A_0(\vec{r})J_0(\vec{r}).
\end{aligned}$$

The factor of $1/2$ accounts for double counting the contributions from pairs of charges.

3.4 Hyper-Surfaces and Conservation of Energy, Momentum and Angular Momentum

The energy and momentum, P^α, in a three-dimensional *hyper-surface* element Ω^γ is

$$P^\alpha_\Omega = \oint_\Omega d\Omega_\gamma T^{\gamma\alpha}. \tag{3.50}$$

Here Ω_γ is a a region of four-space. If $\gamma = 0$ this corresponds to a region with fixed time, and is thus a volume, whereas if $\gamma = 1, 2, 3$ would correspond to a surface area multiplied by some duration in time. More formally,

$$\Omega_\gamma = \int d^4r \partial_\gamma C(r). \tag{3.51}$$

Here, C is some function that is unity in some region and zero outside. For instance, $C = \Theta(r_0 - t)$ would correspond to a hyper-surface (volume in this case) at fixed time t. Only the $\gamma = 0$ component would then be non-zero. If one chose $C = \Theta(r_x - x)$, the hypersurface would be an area for fixed x multiplied by the the entire time. The vector P^α is the energy or momentum that traverses the hyper-surface Ω. As an example, consider the function

$$C(r) = \Theta(r_0 - t_0)\Theta(t_f - r_0)\Theta(R^2 - r_x^2 - r_y^2 - r_z^2), \quad t_f > t_0. \tag{3.52}$$

This is a sphere that appears at $r_0 = t_0$ and disappears at $r_0 = t_f$. For the contribution from $r_0 = t_0$, one has a contribution $dP^\alpha = d\Omega_0\, T^{0\alpha} = d^3r\, T^{0\alpha}(\vec{r}, t_0)$, which is the energy/momentum that appears in the volume at that time. For $t_0 < r_0 < t_f$ the differential contributions come as $dP^\alpha = dS_i\, dt\, T^{i\alpha}(|\vec{r}| = R, t)$, and represent/energy and momentum that flows in/out of the sphere during that time. Finally, the contribution for $r_0 = t_f$ is $dP^\alpha = -d^3rT^{0\alpha}(\vec{r}, t_f)$ is the loss of the remaining energy/momentum once the sphere disappears. The sum of these components must be zero, which can be seen by the divergence theorem in four dimensions,

$$\begin{aligned} \oint d\Omega_\alpha T^{\alpha\beta} &= \int d^4r(\partial_\alpha C(r))T^{\alpha\beta} \\ &= -\int d^4rC(r)\partial_\alpha T^{\alpha\beta} = 0. \end{aligned} \tag{3.53}$$

Thus, stress energy tensor element $T^{\alpha\beta}$ represents the flow of momentum P^α through the hypersurface element $d\Omega_\beta$. Similarly the current J^β represents the flow of charge through the hypersurface element $d\Omega_\beta$.

This also works for angular momentum. As stated earlier the angular momentum is $L^{\alpha\beta} = r^\alpha P^\beta - P^\alpha r^\beta$. The angular momentum flux tensor will will require an additional component to represent the angular momentum that travels through a hyper-surface element $d\Omega_\alpha$. Thus, we define

$$\begin{aligned} M^{\alpha\beta\gamma} &= r^\alpha T^{\beta\gamma} - r^\beta T^{\alpha\gamma}, \\ dL^{\alpha\beta} &= M^{\alpha\beta\gamma} d\Omega\gamma. \end{aligned} \tag{3.54}$$

The choice for M is motivated by the fact that if Ω is purely time-like, $d\Omega_0 = d^3r$, then $M^{\alpha\beta 0}$ indeed looks like the angular momentum density. One can also check this further by testing whether $\partial_\gamma T^{\alpha\beta\gamma} = 0$, which should be true for local conservation of angular momentum,

$$\begin{aligned} \partial_\gamma M^{\alpha\beta\gamma} &= \partial_\gamma (r^\alpha T^{\beta\gamma} - r^\beta T^{\alpha\gamma}) \\ &= (\partial_\gamma r^\alpha)T^{\beta\gamma} - (\partial_\gamma r^\beta)T^{\alpha\gamma} \\ &= g_\gamma^\alpha T^{\beta\gamma} - g_{\gamma\beta} T^{\alpha\gamma} \\ &= T^{\beta\alpha} - T^{\alpha\beta} = 0. \end{aligned} \tag{3.55}$$

In fact, if one believes in angular momentum conservation, this is proof that the stress-energy tensor is symmetric.

3.5 Homework Problems

1. Consider a sphere of radius R and charge Q, where the charge is spread uniformly throughout the sphere.

   (a) Find the strength of the electric field as a function of r.

   (b) Find the electric potential as a function of r.

   (c) Find the potential energy required to move the charges to their positions,

   $$PE = \frac{1}{2}\int d^3r\ \rho(r)V(r).$$

   (d) Find the energy contained in the electric fields.

2. Beginning with $F^{\alpha\beta}$ written in term of $\vec{E}$ and $\vec{B}$, restate

   $$\begin{aligned}\partial_\alpha F^{\alpha\beta} &= 4\pi J^\beta \\ \partial_\alpha \tilde{F}^{\alpha\beta} &= 0\end{aligned}$$

   as

   $$\begin{aligned}\nabla\cdot\vec{E} &= 4\pi\rho, \\ \nabla\times\vec{B} &= \partial_t\vec{E} + 4\pi\vec{J}, \\ \nabla\cdot\vec{B} &= 0, \\ \nabla\times\vec{E} &= -\partial_t\vec{B}.\end{aligned}$$

3. For a charge-free region, $J^\alpha = 0$

   (a) Use Maxwell's equations to write a wave equation for $\vec{E}$, and show the speed of propagation is unity (c).

   (b) For a wave traveling in the $\hat{z}$ direction with the electric field in the $\pm\hat{x}$ direction, write a solution for the propagating plane wave for both $\vec{E}(\vec{r},t)$ and $\vec{B}(\vec{r},t)$.

4. First calculate $\hbar c$ in standard mks units. Then, using the fact that the charge on an electron is 1.602×10^{-19} Coulombs, find the constant k in mks units used in Coulomb's law, $PE = kq^2/r$. Use the fact that $PE = e^2/r$, where $e^2 = \hbar c/137.036$.

5. Consider two very large parallel capacitor plates of area A, carrying charge densities σ and $-\sigma$, and oriented perpendicular to the z axis. The plates are initially at a very small separation at $t=0$, but are pulled apart, moving with constant non-relativistic velocities $v/2$ and $-v/2$.

   (a) What is the electric field between the plates?

   (b) Find all four non-zero elements of the stress-energy tensor (T_{xx}, T_{yy}, T_{zz} and T_{00}). Check that the stress-energy tensor is traceless.

(c) In hydrodynamics, the work done by an expanding a gas is PdV. Here, because the expansion is along the z axis the work is $T_{zz}dV$. What is the power required to pull the plates apart at these velocities?

(d) What is energy density of the field between the plates?

(e) What is the rate (energy per time) at which the field energy between the plates increases due to the growing volume?

4 Electrostatics

Here, we consider the electric field of fixed charge distributions. All currents are set to zero, so there is only electric field, and all time derivatives in Maxwell's equations are neglected.

4.1 Gauss's Law

Beginning with

$$\nabla \cdot \vec{E} = 4\pi\rho, \tag{4.1}$$

one can integrate over a volume, then use the divergence theorem (also known as Gauss's theorem) to find Gauss's law

$$\begin{aligned} \oint d^3r\; \nabla \cdot \vec{E}(\vec{r}) &= 4\pi \oint d^3r\; \rho(\vec{r}), \qquad (4.2)\\ \oint d\vec{A} \cdot \vec{E} &= 4\pi Q. \qquad (4.3) \end{aligned}$$

For a point charge, define the volume as a sphere of radius r surrounding the charge Q, one then quickly finds Coulomb's law.

$$\begin{aligned} 4\pi r^2 E_r &= 4\pi Q, \qquad (4.4)\\ E_r &= \frac{Q}{r^2}. \qquad (4.5) \end{aligned}$$

4.2 Potential Energy of a Fixed Continuous Charge Distribution

For a continuous charge density $\rho(\vec{r})$, the potential energy required to bring the last bit of charge δQ to a position r is

$$\begin{aligned} \delta PE &= \Phi(\vec{r})\delta Q, \quad \Phi = A_0, \qquad (4.6)\\ &= \int d^3r'\; \rho(\vec{r}') \frac{\delta Q}{|\vec{r} - \vec{r}'|}. \end{aligned}$$

It would be tempting to write the entire potential energy as a sum over all δQ, or as a separate integral of $\int d^3r\; \rho(\vec{r})$, but that would lead to double counting. The double counting would come from considering the effect of brining a differential charge $\delta Q = d^3r\rho(\vec{r})$ towards each differential charge $\delta Q' = d^3r'\rho(\vec{r}')$, and the opposite. Thus, one introduces a factor of $1/2$ when writing the entire potential energy,

$$\begin{aligned} PE &= \frac{1}{2}\int d^3r d^3r'\; \frac{\rho(\vec{r})\rho(\vec{r}')}{|\vec{r} - \vec{r}'|} \qquad (4.7)\\ &= \frac{1}{2}\int d^3r\; \rho(\vec{r})\Phi(\vec{r}). \qquad (4.8) \end{aligned}$$

Example 4.1:

Find the net potential energy for a charge Q uniformly spread out in a sphere of radius R.

First, find the potential $\Phi(r)$. For $r > R$, it is easy, $\Phi = Q/r$. For $r < R$, you need to first find the electric field. Beginning with Gauss's law,

$$\begin{aligned}
E &= \frac{Qr^3/R^3}{r^2} = \frac{Qr}{R^3}, \\
\Phi &= \frac{Q}{R} + \int_r^R drE(r) \\
&= \frac{Q}{R} + \frac{1}{2}\frac{Q}{R^3}(R^2 - r^2) \\
&= \frac{3Q}{2R} - \frac{Qr^2}{2R^3}.
\end{aligned}$$

Next, integrate over the charge density, $3Q/4\pi R^3$, to get the potential energy,

$$\begin{aligned}
PE &= \frac{1}{2}\frac{3Q}{4\pi R^3}\int_0^R 4\pi r^2 dr \left[\frac{3Q}{2R} - \frac{Qr^2}{2R^3}\right] \\
&= \frac{3Q^2}{5R}.
\end{aligned}$$

4.3 Laplace's Equations and a Fixed Point Charge

First, we consider expressions, mainly for the electric potential $\Phi = A_0$, where nothing changes with time. In that case, Maxwell's equation all terms with ∂_t*Something* are set to zero,

$$\begin{aligned}
\nabla \cdot \vec{E} &= 4\pi\rho, \\
\vec{E} &= -\nabla\Phi, \\
\nabla^2\Phi &= -4\pi\rho.
\end{aligned} \tag{4.9}$$

If there is no charge density one is left with Laplace's equation,

$$\nabla^2\Phi = 0. \tag{4.10}$$

This is applicable for any region with no charge density, and if charges exist outside the region, one must solve Laplace's equations with boundary conditions.

The most obvious example of a field free region, with a charge confined outside, is that of a point charge Q at the origin. In that case $\rho = 0$ for $r > \epsilon$. Gauss's law, combined with symmetry gives

$$\begin{aligned}
4\pi R^2|\vec{E}| &= 4\pi Q, \\
\vec{E} &= \frac{Q}{r^2}\hat{r}.
\end{aligned} \tag{4.11}$$

The potential must then be

$$\Phi = \frac{Q}{r}. \tag{4.12}$$

From this constraint, the identity

$$\nabla^2 \left(\frac{1}{r}\right) = -4\pi\delta(\vec{r}) \tag{4.13}$$

becomes manifest. Thus, the function Q/r is a solution to Laplace's equation in the region $r > \epsilon$ satisfying the boundary condition that the electric flux entering the region is $4\pi Q$.

4.4 Laplace's Equations in Cartesian Coordinates

In Cartesian coordinates separation of variables assumes that Φ is a product of three pieces,

$$\begin{aligned} \Phi(x,y,z) &= \psi_x(x)\psi_y(y)\psi_z(z) \\ \partial_x^2\psi(x) &= -k_x^2\psi(x), \\ \partial_y^2\psi(y) &= -k_y^2\psi(y), \\ \partial_z^2\psi(z) &= -k_z^2\psi(z), \\ \nabla^2\Phi &= -(k_x^2+k_y^2+k_z^2)\Phi = 0, \\ k_x^2+k_y^2+k_z^2 &= 0. \end{aligned} \tag{4.14}$$

With these eigenvalues,

$$\begin{aligned} \psi_x(x) &= A_x e^{ik_x x} + B_x e^{-ik_x x}, \\ \psi_y(y) &= A_x e^{ik_y y} + B_x e^{-ik_y y}, \\ \psi_z(z) &= A_z e^{ik_z z} + B_z e^{-ik_z z}, \end{aligned} \tag{4.15}$$

with A_i and B_i being arbitrary constants chosen to fit the boundary conditions. Because $k_x^2 + k_y^2 + k_z^2 = 0$, at least one of the wave numbers must be complex.

4.5 Laplace's Equations and Solutions in Spherical Coordinates

To obtain Laplace's equation in spherical coordinates, we first write the gradient operator in those coordinates,

$$\nabla = \hat{r}\frac{dr}{d\ell_r}\partial_r + \hat{\theta}\frac{d\theta}{d\ell_\theta}\partial_\theta + \hat{\phi}\frac{d\phi}{d\ell_\theta}\partial_\phi, \tag{4.16}$$

where the small steps $d\ell_i$ represent a Cartesian coordinate system in those directions. The geometry,

$$d\ell_r = dr,\ \ d\ell_\theta = r d\theta,\ \ d\ell_\phi = r\sin\theta, \tag{4.17}$$

gives

$$\nabla = \hat{r}\partial_r + \hat{\theta}\frac{1}{r}\partial_\theta + \hat{\phi}\frac{1}{r\sin\theta}\partial_\phi. \tag{4.18}$$

Next, before finding the Laplacian, $\nabla \cdot \nabla$, we realize that the unit vectors themselves depend on the angle. Thus, when one takes the second round of derivatives w.r.t. θ and ϕ, one must remember that $\hat{\theta}$ and $\hat{\phi}$ depend on angle. For small changes in the angles

$$\begin{aligned}
\hat{r} &= \hat{r}_0 + \hat{\theta}_0\delta\theta + \hat{\phi}_0\sin\theta\delta\phi, \qquad (4.19)\\
\hat{\theta} &= \hat{\theta}_0 - \hat{r}_0\delta\theta + \hat{\phi}_0\cos\theta\delta\phi,\\
\hat{\phi} &= \hat{\phi}_0 - \hat{r}_0\sin\theta\delta\phi - \hat{\theta}_0\cos\theta\delta\phi.
\end{aligned}$$

Although we will set $\delta\theta$ and $\delta\phi$ to zero, that will only be after taking the second divergence. The gradient operator is then

$$\begin{aligned}
\nabla &= \hat{r}_0\left(\partial_r - \delta\theta\frac{1}{r}\partial_\theta - \delta\phi\frac{\cos\theta}{r\sin\theta}\partial_\phi\right) \qquad (4.20)\\
&+\hat{\theta}_0\left(\frac{1}{r}\partial_\theta + \delta\theta\partial_r - \delta\phi\frac{\cos\theta}{r\sin\theta}\partial_\theta\right)\\
&+\hat{\phi}_0\left(\frac{1}{r\sin\theta}\partial_\phi + \delta\phi\sin\theta\partial_r + \delta\phi\cos\theta\frac{1}{r}\partial_\theta\right).
\end{aligned}$$

We will set $\delta\theta = \delta\phi = 0$ after taking the second round of derivatives, so we need only worry about the term with $\delta\theta$ term in the part proportional $\hat{\theta}$ and the $\delta\phi$ term in the part proportional to $\hat{\phi}$ because $\partial_\theta\delta\theta = \partial_\phi\delta\phi = 1$ while all others are zero. Thus,

$$\begin{aligned}
\nabla\cdot\nabla &= \partial_r^2 + \frac{1}{r^2}\partial_\theta^2 + \frac{1}{r^2\sin^2\theta}\partial_\phi^2 + \frac{1}{r}\partial_r + \frac{1}{r\sin\theta}\left(\sin\theta\partial_r + \cos\theta\frac{1}{r}\partial_\theta\right) \qquad (4.21)\\
&= \partial_r^2 + \frac{2}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2 + \frac{\cos\theta}{r^2\sin\theta}\partial_\theta + \frac{1}{r^2\sin^2\theta}\partial_\phi^2,\\
\nabla^2 &= \frac{1}{r^2}\partial_r(r^2\partial_r) + \frac{1}{r^2\sin\theta}\partial_\theta(\sin\theta\partial_\theta) + \frac{1}{r^2\sin^2\theta}\partial_\phi^2. \qquad (4.22)
\end{aligned}$$

Next, to solve $\nabla^2\Phi = 0$, we first write Φ as a product of three pieces,

$$\Phi(r,\theta,\phi) = R(r)P(\theta)Q(\phi). \tag{4.23}$$

If one assumes that each function satisfies the following differential equations,

$$\begin{aligned}
\partial_\phi^2 Q_m(\phi) &= -m^2\phi, \qquad (4.24)\\
\frac{1}{r^2}\partial_r(r^2\partial_r R) &= \lambda R,\\
\frac{1}{\sin\theta}\partial_\theta(\sin\theta P_{\lambda m}(\theta)) - \frac{m^2}{\sin^2\theta}P_{\lambda m}(\theta) &= -\lambda P_{\lambda m}(\theta)
\end{aligned}$$

Laplace's equations will be satisfied. Because the function must be periodic, m is an integer. The solutions $R(r)$ behave as r^n and $\lambda = n(n+1)$. For positive values, $n = \ell$, one finds the same

λ as for $n = -\ell - 1$. Thus, we switch labels from λ to ℓ, with $\lambda = \ell(\ell+1)$, and one can write the general radial solutions as

$$\begin{aligned} R_\ell(r) &= Ar^\ell + Br^{-\ell-1}, \\ \frac{1}{\sin\theta}\partial_\theta(\sin\theta\partial_\theta P_{\ell m}(\theta)) &= (-\ell(\ell+1)+m^2)P_{\ell m}(\theta). \end{aligned} \tag{4.25}$$

The functions $P_{\ell,m}(\theta)$ are known as *associated Legendre polynomials*, and the products

$$\begin{aligned} Y_{\ell,m}(\theta,\phi) &= P_{\ell,m}(\theta)e^{im\phi}, \\ \frac{1}{\sin\theta}\partial_\theta(\sin\theta Y_{\ell,m}(\theta)) + \frac{1}{\sin^2\theta}\partial_\phi^2 Y_{\ell,m}(\theta,\phi) &= \ell(\ell+1)Y_{\ell,m}(\theta,\phi). \end{aligned} \tag{4.26}$$

are referred to as *spherical harmonics*. The functions are orthonormal,

$$\int d\phi\, d\cos\theta\, Y_{\ell,m}(\theta,\phi)Y_{\ell',m'}(\theta,\phi) = \delta_{\ell\ell'}\delta_{mm'}. \tag{4.27}$$

Recurrence relations allow one to generate solutions for $Y_{\ell m}$ for a given ℓ and m from solutions for lower ℓ or m. The operators, L_+ and L_- change $P_{\ell,m}$ to $P_{\ell,m\pm1}$ and are known as raising and lowering operators,

$$\begin{aligned} L_\pm &= -e^{\pm i\phi}\left(\pm i\partial_\theta - \cot\theta\partial_\phi\right), \\ L_\pm P_{\ell m\pm1}(\theta,\phi) &= \left[\ell(\ell+1) - m(m\pm1)\right]P_{\ell m}(\theta\phi). \end{aligned} \tag{4.28}$$

Checking this relation is a homework problem. One can see that because both ℓ and m are integers that $|m| \le \ell$, i.e. there $2\ell+1$ values of m for each ℓ, from $m = -\ell$ to $m = \ell$. Simple expansions provide the form $P_{\ell m=0}$. Beginning with the definition of Legendre polynomials,

$$P_\ell(\cos\theta) \equiv \frac{1}{\sqrt{2\ell+1}}P_{\ell,m=0}(\theta,\phi), \tag{4.29}$$

one can express

$$\begin{aligned} \partial_x\left[(1-x^2)\partial_x P_\ell(x)\right] &= -\ell(\ell+1)P_\ell(x), \\ P_\ell(x=\cos\theta) &= \frac{1}{2^n}\sum_{k=0}^{\ell}\left(\frac{\ell!}{(\ell-k)!k!}\right)^2(x-1)^{\ell-k}(x+1)^k, \\ P_\ell(\cos\theta) &= \sqrt{\frac{4\pi}{2\ell+1}}Y_{\ell m=0}(\theta). \end{aligned} \tag{4.30}$$

One can then use the raising and lowering operators to find expressions for $Y_{\ell m}$ for any m.

Various $Y_{\ell m}(\theta, \phi)$:

$$\begin{aligned}
Y_{0,0} &= \frac{1}{\sqrt{4\pi}}, \\
Y_{1,0} &= \sqrt{\frac{3}{4\pi}} \cos\theta, \\
Y_{1,\pm 1} &= \mp\sqrt{\frac{3}{8\pi}} \sin\theta e^{i\pm\phi}, \\
Y_{2,0} &= \sqrt{\frac{5}{16\pi}}(3\cos^2\theta - 1), \\
Y_{2,\pm 1} &= \mp\sqrt{\frac{15}{8\pi}} \sin\theta\cos\theta e^{\pm i\phi}, \\
Y_{2,\pm 2} &= \sqrt{\frac{15}{32\pi}} \sin^2\theta e^{\pm 2i\phi}.
\end{aligned} \tag{4.31}$$

Various $P_\ell(x)$:

$$\begin{aligned}
P_0(x) &= 1, \\
P_1(x) &= x, \\
P_2(x) &= \frac{1}{2}(3x^2 - 1), \\
P_3(x) &= \frac{1}{2}(5x^3 - 3x), \\
P_4(x) &= \frac{1}{8}(35x^4 - 30x^2 + 3).
\end{aligned} \tag{4.32}$$

For negative m, one can use the identity,

$$Y_{\ell -m}(\theta, \phi) = (-1)^m Y^*_{\ell m}(\theta, \phi). \tag{4.33}$$

Legendre polynomials satisfy a number of identities,

$$\begin{aligned}
P_\ell(x = 1) &= 1, \\
\int_{-1}^{1} dx\, P_\ell(x) P_{\ell'}(x) &= \frac{2}{2\ell + 1}\delta_{\ell\ell'}, \\
\sum_\ell (2\ell + 1) P_\ell(x) P_\ell(x') &= 2\delta(x - x'), \\
(2\ell + 1) P_\ell(x) &= \frac{d}{dx}\left[P_{\ell+1}(x) - P_{\ell-1}(x)\right], \\
(\ell + 1) P_{\ell+1}(x) &= (2\ell + 1) x P_\ell(x) - \ell P_{\ell-1}(x), \\
P_\ell(x) &= \frac{1}{2^\ell \ell!}\frac{d^\ell}{dx^\ell}(x^2 - 1)^\ell \ \text{(Rodriguez formula)}, \\
\frac{1}{\sqrt{1 - 2xt + t^2}} &= \sum_\ell P_\ell(x) t^\ell \ \text{(generating function)}.
\end{aligned} \tag{4.34}$$

4.6 Solutions to Laplace's Equations in Cylindrical Coordinates

In cylindrical coordinates, $\rho \equiv \sqrt{x^2 = y^2}$, $\phi \equiv \tan^{-1} y/x$ and z, Laplace's equation becomes

$$\begin{aligned} \nabla^2\Phi &= \left\{\frac{1}{\rho}\partial_\rho(\rho\partial_\rho)\Phi + \frac{1}{\rho^2}\partial_\phi^2 + \partial_z^2\right\}\Phi = 0, \\ \Phi(\rho,\phi,z) &= R(\rho)Q(\phi)Z(z). \end{aligned} \tag{4.35}$$

First, choose Q and Z to be eigenstate of the corresponding part of the Laplacian,

$$\begin{aligned} \partial_z^2 Z(z) &= k_z^2 Z(z), \\ \partial_\phi^2 Q(\phi) &= -m^2 Q(\phi). \end{aligned} \tag{4.36}$$

With these choices,

$$\begin{aligned} Z(z) &= Ae^{k_z z} + Be^{-k_z z}, \\ Q(\phi) &= e^{im\phi}. \end{aligned} \tag{4.37}$$

The equation for $R(\rho)$, which depends on m and k_z so it is labeled $R_m(k_z,\rho)$, is then

$$\begin{aligned} \left(\frac{1}{\rho}\partial_\rho(\rho\partial_\rho) - \frac{m^2}{\rho^2} + k_z^2\right)R_m(k_z,\rho) &= 0, \\ \left(\partial_\rho^2 + \frac{1}{\rho}\partial_\rho\right)R_m(k_z,\rho) + \left(k_z^2 - \frac{m^2}{\rho^2}\right)R_m(k_z,\rho) &= 0. \end{aligned} \tag{4.38}$$

If one makes the change of variables, and considers for $R(x = k_z\rho)$, the differential equation becomes

$$\left(\partial_x^2 R_m(x) + \frac{1}{x}\partial_x\right)R_m(x) + \left(1 - \frac{m^2}{x^2}\right)R_m(x) = 0. \tag{4.39}$$

For the solution to have a form

$$R_m(x) = x^\alpha \sum_{j=0}^{\infty} a_j x^2, \tag{4.40}$$

the lowest power of in the expansion, α, must either be $|m|$ or $-|m|$. This can be seen by letting $k \to 0$ in Eq. (4.38), so that in Eq. (4.39) one can simplify the equation by replacing $(1 - m^2/x^2)$ with $1/x^2$ (see the end-of-chapter problem).

The solutions with each power are known as $J_m(x)$ and $N_m(x)$, with it being understood that the index is positive. The general solution has arbitrary constants A and B which will be determined by boundary conditions,

$$R_m(x) = AJ_m(x) + BN_m(x). \tag{4.41}$$

As stated above, for small x and $m > 0$,

$$\begin{aligned} J_m(x) &\sim x^m + \cdots, \\ N_m(x) &\sim x^{-m} + \cdots, \end{aligned} \tag{4.42}$$

where the functions J and N are known as Bessel functions or Neumann functions respectively. The number m is known as the order of the equation. From the differential, one can see quickly derive the recursion relation for the coefficients a_m in Eq. (4.40),

$$a_{2j} = -\frac{1}{4j(j+\alpha)} a_{2j-2}, \tag{4.43}$$

with all the odd coefficients vanishing.

The large and small x expansions are:

$$\begin{aligned} x \ll 1, \qquad J_m(x) &\rightarrow \frac{1}{\Gamma(m+1)} \left(\frac{x}{2}\right)^m, \\ N_m(x) &\rightarrow \begin{cases} \frac{2}{\pi}\left[\ln\left(\frac{x}{2}\right) + 0.5772\cdots\right], & m = 0 \\ -\frac{\Gamma(m)}{\pi}\left(\frac{2}{x}\right)^m, & m \neq 0. \end{cases}, \end{aligned} \tag{4.44}$$

and the expansion for large x are:

$$\begin{aligned} x \gg 1, m \qquad J_m(x) &\rightarrow \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{m\pi}{2} - \frac{\pi}{4}\right), \\ N_m(x) &\rightarrow \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{m\pi}{2} - \frac{\pi}{4}\right). \end{aligned} \tag{4.45}$$

Each of these expressions assumes $m \geq 0$, and the constant 0.57772 is Euler's constant.

4.7 Boundary Value Problems

Boundary value problems involve finding solutions for Laplace's equations that satisfy the B.C. for some region of space. The B.C. must be satisfied at all boundaries of the space. Often, the boundaries are either a conductor, constant potential, or at infinity, with the potential either vanishing or behaving with a known manner, e.g. becoming a constant electric field. If the potential is defined at the boundary, this is known as a Dirichlet problem. Another option would be to define the electric field or charge density at a boundary, or one can have some mixture.

4.7.1 Method of Images

This method can be applied in certain situation where charges are in the presence of conducting surfaces that divide space into separate regions. Conducting surfaces have equipotential, and if one is considering a given sub-space (e.g. the region above an infinite conducting plane), one can attempt to consider how one could mimic the effect of the conducting surface by placing charges in the other spaces (e.g. below the plane). The typical example is to consider a charge $+Q$ above an infinite grounded conducting plane defined by $z = 0$ at a postition $x = y = 0, z = a$. For the $z > 0$ region, one could consider an imaginary charge $-Q$ at a position $x = y = 0, z = -a$. Clearly, this imaginary charge would lead to $\Phi(x, y, z = 0+) = 0$. Thus, the overall potential for $z > 0$ is

$$\Phi(x, y, z > 0) = \frac{Q}{|\vec{r} - a\hat{z}|} - \frac{Q}{|\vec{r} + a\hat{z}|}. \tag{4.46}$$

For $z < 0$ the conducting plane shields the effect of the charge in the upper half plane, and $\Phi(x, y, z < 0) = 0$.

Charges are attracted to their images, thus a charge is attracted to a nearby conductor. This is often relevant to accelerator design.

Example 4.2:

Consider a point charge $+Q$ outside a grounded conducting sphere. The sphere has radius R and is centered at the origin, and the point charge is at position $a\hat{z}$.

Solution:
We will consider an image charge Q_i place inside the sphere at position $x_i = y_i = 0, z_i = a_i$, so that the sum of the two potentials cancel at $r = R$,

$$\begin{aligned} \Phi(r = R, \theta) &= \frac{Q}{\sqrt{R^2 \sin^2\theta + (a - R\cos\theta)^2}} + \frac{Q_i}{\sqrt{R^2 \sin^2\theta + (a_i - R\cos\theta)^2}} \\ &= \frac{Q}{R\sqrt{1 + a^2/R^2 - 2(a/R)\cos\theta}} + \frac{Q_i}{R\sqrt{1 + a_i^2/R^2 - 2(a_i/R)\cos\theta}} \end{aligned} \tag{4.47}$$

In order for the potential to vanish for all $\cos\theta$,

$$\frac{Q}{\sqrt{1 + a^2/R^2}} = -\frac{Q_i}{\sqrt{1 + a_i^2/R^2}}, \tag{4.48}$$

$$\frac{2a/R}{1 + a^2/R^2} = \frac{2a_i/R}{1 + a_i^2/R^2}. \tag{4.49}$$

The latter expression becomes a quadratic equation with two solutions, $a_i = a$ and $a_i = R^2/a$. The first solution is obvious – the image charge sits right on top of the of the real charge, but is in the same region, so we neglect it. The second solution is the one we desire, and solving for Q_i,

$$\begin{aligned} Q_i &= -Q\sqrt{\frac{1 + a_i^2/R^2}{1 + a^2/R^2}} \\ &= -Q\sqrt{\frac{1 + R^2/a^2}{1 + a^2/R^2}} \\ &= -\frac{QR}{a}. \end{aligned} \tag{4.50}$$

The potential is zero inside the sphere.

4.7.2 Boundary Value Problems Using Cartesian Solutions to the Laplace Equation

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Example 4.3:

a) Consider an infinite plane defined by $z = 0$, where the potential has the form

$$\Phi(x, y, z = 0) \;=\; V_0 \cos(qx). \tag{4.51}$$

Assuming the remainder of the volume is vacuum, find the potential for all $x, y, z \neq 0$.
Solution:
To match the B.C. at $z = 0$, and to decay to zero for large z, the solution is

$$\Phi(x, y, z) \;=\; V_0 \cos(qx) e^{-q|z|}. \tag{4.52}$$

4.7.3 Boundary Value Problems Using Spherical Harmonics

Laplace's equation is applicable in any charge-free region, but doesn't mean it doesn't apply in problems with charge densities. You simply only use Laplace's equation in the charge-free part of the volume. There are a few, with emphasis on *few*, boundary-value problems one can easily perform using the spherical harmonics mentioned before. The most common non-trivial example is a conducting sphere in a constant electric field, which we work out below.

Example 4.4:

A conductor of radius R is placed at the origin in a field which previously was uniform, $\vec{E} = E_0\hat{z}$.

1. Find the potential for $r > R$. This region has no charges, hence it satisfies Laplace's equations. For the solutions that behave as r^ℓ, only the $\ell = 1$ solution can appear, because otherwise the solution would not look like a constant field at large r. Because a conductor's potential has to be constant at $r = R$, only the $\ell = 1$ term of the $1/r^{\ell+1}$ solutions can be non-zero, because they need to cancel the r^ℓ solutions to for all θ. Thus, the solution must be of the form,

$$\begin{aligned}
\Phi &= \cos\theta\left\{-E_0 r + \frac{B}{r^2}\right\}, \quad r > R, \\
E_0 R &= \frac{B}{R^2}, \\
B &= E_0 R^3, \\
\Phi(r > R) &= -E_0\cos\theta\left\{r - \frac{R^3}{r^2}\right\}.
\end{aligned}$$

   One can always add an arbitrary constant to the potential, which would correspond to the $\ell = 0$ term.

2. Find the electric field for all r.

$$\begin{aligned}
\vec{E} &= -\nabla\Phi \\
&= E_0\hat{z} - \hat{r}\partial_r\Phi - \hat{\theta}\frac{1}{r}\partial_\theta\Phi \\
&= E_0\hat{z} + \hat{r}\cos\theta\left(\frac{2E_0R^3}{r^3}\right) + \hat{\theta}\sin\theta\left(\frac{E_0R^3}{r^3}\right) \\
&= \hat{r}E_0\cos\theta\left(1+\frac{2R^3}{r^3}\right) + \hat{\theta}E_0\sin\theta\left(\frac{R^3}{r^3}-1\right).
\end{aligned}$$

The electric field must be perpendicular to a conducting surface, because it is at constant potential, which is indeed satisfied by seeing that the $\hat{\theta}$ component vanishes at $r = R$.

3. Find the charge density per unit area on the surface of the sphere. Because there is no electric field inside the conductor, one can consider a small area dA on the sphere at angle θ. Gauss's law relates the radial part of the electric field to the charge density σ,

$$\begin{aligned}
E_r dA &= 4\pi\sigma dA, \\
3E_0\cos\theta &= 4\pi\sigma, \\
\sigma &= \frac{3E_0}{4\pi}\cos\theta.
\end{aligned}$$

4.7.4 Boundary Value Problems Using Cylindrical Harmonics

This is very similar in spirit to the spherical case.

Example 4.5:

Consider a long conducting cylinder of radius R that is positioned so that its axis is perpendicular to an initially uniform electric field, $\vec{E} = E_0\hat{x}$. Assuming the axis of the cylinder is defined by $z = 0$, find the electric potential and field at all points outside the conductor.

Solution:

The initial field is

$$\Phi = -E_0\rho\cos\phi. \tag{4.53}$$

Since we only wish to add corrections with the correct ϕ dependence, we assume the answer to be of the form

$$\Phi(\rho,\phi) = -E_0\rho\cos\phi + AJ_1(k_z\rho)\cos(\phi) + BN_1(k_z\rho)\cos\phi, \quad k_z \to 0. \tag{4.54}$$

Because $k_z \to 0$, we need only consider the first order term when expanding in x. In that case, N_1 expands as $1/\rho$ and J_1 expands as ρ. We can neglect the term that expands as ρ because it was already included by choosing the long-range field. Thus

$$\Phi(\rho,\phi) = -E_0\rho\cos\phi + \frac{C}{\rho}\cos\phi, \quad k_z \to 0. \tag{4.55}$$

By inspection, to cancel the potential at $\rho = R$, $C = R^2 E_0$,

$$\begin{aligned}
\Phi(\rho,\phi) &= \left(-E_0\rho + E_0\frac{R^2}{\rho}\right)\cos\phi, \qquad (4.56)\\
&= -E_0 x + \frac{E_0 x R^2}{\rho^2}\\
E_x &= E_0 - \frac{E_0 R^2}{\rho^2} + \frac{E_0 x^2 R^2}{\rho^4},\\
E_y &= \frac{E_0 x y R^2}{\rho^4}.
\end{aligned}$$

4.8 Solving Boundary Value Problems Numerically

Typically, boundary value problems are multidimensional second-order differential equations (Laplace's equation). Thus, they can be difficult to solve given that you need to know two layers of BC to integrate forward to the third layer, whereas boundary conditions are typically provided only for one layer. One strategy for solving such problems is to look for iterative solutions. Assume you have a guess of the solution, $\Phi_0(x,y,z)$, where x, y and z are discrete values. For example you might divide the x range, from $x_{\rm min}$ to $x_{\rm max}$ into $N+1$ values of size $dx = (x_{\rm max} - x_{\rm min})/N$. In the discretized space, Laplace's equation becomes,

$$\begin{aligned}
\partial_x^2\Phi + \partial_y^2\Phi + \partial_z^2\Phi &= 0, \qquad (4.57)\\
\frac{1}{dx}\left(\frac{\Phi(x+dx,y,z)-\Phi(x,y,z)}{dx} - \frac{\Phi(x,y,z)-\Phi(x-dx,y,z)}{dx}\right)&\\
+\frac{1}{dy}\left(\frac{\Phi(x,y+dy,z)-\Phi(x,y,z)}{dy} - \frac{\Phi(x,y,z)-\Phi(x,y-dy,z)}{dy}\right)&\\
+\frac{1}{dz}\left(\frac{\Phi(x,y,z+dz)-\Phi(x,y,z)}{dz} - \frac{\Phi(x,y,z)-\Phi(x,y,z-dz)}{dz}\right)&\\
\frac{1}{dx^2}\left(\Phi(x+dx,y,z) - 2\Phi(x,y,z) + \Phi(x-dx,y,z)\right)&\\
+\frac{1}{dy^2}\left(\Phi(x,y+dy,z) - 2\Phi(x,y,z) + \Phi(x,y-dy,z)\right)&\\
\frac{1}{dz^2}\left(\Phi(x,y,z+dz) - 2\Phi(x,y,z) + \Phi(x,y,z-dz)\right) &= 0.
\end{aligned}$$

One can solve for $\Phi(x,y,z)$ in terms of its neighbors,

$$\begin{aligned}
\Phi(x,y,z) &= \frac{1}{2}\frac{(dx)^2(dy)^2(dz)^2}{(dx)^2(dy)^2+(dy)^2(dz)^2+(dx)^2(dz)^2} \qquad (4.58)\\
&\cdot\left(\frac{1}{(dx)^2}[\Phi(x+dx,y,z)+\Phi(x-dx,y,z)]\right.\\
&+\frac{1}{(dy)^2}[\Phi(x,y+dy,z)+\Phi(x,y-dy,z)]\\
&\left.+\frac{1}{(dz)^2}[\Phi(x,y,z+dz)+\Phi(x,y,z-dz)]\right).
\end{aligned}$$

If $dx = dy = dz$, this is simply an average of the six neighbors.

To perform the iterative solution, one finds new values $\Phi(x, y, z)$ for each x, y, z from Eq. (4.58) using $\Phi_0(x, y, z)$ on the right-hand side of the equation. If the solution is correct, Φ will not differ from Φ_0. However, if Φ_0 is incorrect, it will differ. If the iterative procedure indeed is convergent, the new value of Φ will be closer to the answer than Φ_0, and if one repeats the procedure numerous times, replacing Φ_0 with the new answer at each iteration, Φ will approach the correct answer.

4.9 Homework Problems

1. Consider two charge densities,

$$\begin{aligned}\rho_1(\vec{r}) &= \frac{3Q_1}{4\pi R_1^3}\Theta(R_1 - |\vec{r}|),\\ \rho_2(\vec{r}) &= \frac{3Q_2}{4\pi R_2^3}\Theta(R_2 - |\vec{r} - a\hat{x}|).\end{aligned}$$

   (a) Find the potential energy of the charge distribution when $a \to \infty$.

   (b) Find the change in the potential energy for moving from $a = \infty$ to $a > R_1 + R_2$ but finite.

2. Consider two concentric conducting spherical shells of radius R and $R + a$. The charges on the spheres are Q and $-Q$.

   (a) Calculate the capacitance of the spheres, $C = Q/V$.

   (b) For a potential V, find the electric field as a function of r for all r.

   (c) For a potential V calculate the net energy stored in the electric field.

   (d) Compare this to $CV^2/2$.

3. Show that

$$\begin{aligned}Y_{\ell,\ell}(\theta, \phi) &= c_\ell e^{i\ell\phi} \sin^\ell \theta,\\ c_\ell &= \left[\frac{(-1)^\ell}{2^\ell \ell!}\right]\sqrt{\frac{(2\ell+1)(2\ell)!}{4\pi}}.\end{aligned}$$

   is a solution to

$$\frac{1}{\sin\theta}\partial_\theta(\sin\theta\partial_\theta Y(\theta, \phi) + \frac{1}{\sin^2\theta}\partial_\phi^2 Y(\theta, \phi) = -\ell(\ell+1)Y(\theta, \phi),$$

   and that it has unit normalization.

4. Show that the expansion in Eq. (4.30) for Legendre polynomials is indeed a solution for

$$\left[(1 - x^2)\partial_x^2 - 2x\partial_x + \ell(\ell+1)\right] P_\ell(x) = 0.$$

5. Show that the solution to Laplace's equation in cylindrical coordinates when $k_z = 0$,

$$\left(\partial_\rho^2 + \frac{1}{\rho}\partial_\rho\right) R(\rho) - \frac{m^2}{\rho^2} R(\rho) = 0,$$

becomes

$$R_m(\rho) = \begin{cases} A\rho^{-m} + B\rho^m, & m \neq 0, \\ C\ln(\rho), & m = 0 \end{cases}.$$

6. Consider the function $\Phi = (x^2 + y^2) = r^2 \sin^2\theta$.
   (a) Calculate $\nabla^2\Phi$ in Cartesian coordinates.
   (b) Repeat in spherical coordinates

7. Consider a cavity that extends from $x = -a$ to $x = a$ and from $y = 0$ to $y = \infty$, i.e. it is infinite in the y direction. Assume it is infinite in both directions in the z direction. Along the $y = 0$ boundary, the surface is an insulator kept at a uniform potential V_0, while the boundaries along the $x = \pm a$ surfaces are grounded. Note this means that at the corner of the boundaries the potential is discontinuous, thus one might need to imagine an infinitesimal insulator at the intersection of the boundaries.
   (a) Write down general solutions for the system that exponentially die for large y, and that satisfy the B.C. that $\Phi(x = -a, y, z) = \Phi(x = a, y, z) = 0$. For the moment, ignore the B.C. at the $y = 0$ surface.
   (b) Find the sum of such solutions from (a) that satisfies the B.C. that $V(x, y = 0, z) = V_0$.

8. Consider the solution to a point charge outside a conducting sphere of radius R performed with images from the notes. Consider two point charges, one with charge Q at $-a\hat{z}$, and a second with charge $-Q$ at $a\hat{z}$. As $a \to \infty$, the sphere sees a constant electric field, $\vec{E} = 2Q/a^2\hat{z}$. Find the electric potential from the two point charges and from the two images in the limit that a and Q both go to infinity in such a way that $2Q/a^2 = E_0$. Compare your solution to that you get from using spherical harmonics.

9. Consider two infinite conducting planes at $z = a$ and $z = -a$. A point charge Q is replaced at the origin. Find a set of image charges that satisfy the B.C. for $-a < z < a$.

10. Using the solution for a conducting cylinder in a constant field from the notes, show that the electric field is perpendicular to the surface at $\rho = R$.

11. Consider a sphere of radius R centered at the origin, The surface of the potential is $V(\cos\theta)$.
    (a) In spherical coordinates, using the azimuthal symmetry, the potential at $r = R$ can be written as

$$\Phi(r = R, \cos\theta) = \sum_\ell a_\ell P_\ell(\cos\theta).$$

Find C_ℓ in the expression for a_ℓ of the form,

$$a_\ell = C_\ell \int_{-1}^{1} dx\, \Phi(r = R, x) P_\ell(x).$$

Here are some identities you might find useful:

$$\begin{aligned}
P_\ell(x=1) &= 1, \\
\sum_\ell (2\ell+1)P_\ell(x)P_\ell(x') &= 2\delta(x-x'), \\
\int_{-1}^{1} dx\; P_\ell(x)P_{\ell'}(x) &= \frac{2}{2\ell+1}\delta_{\ell\ell'}, \\
\sum_\ell (2\ell+1)P_\ell(x)P_\ell(x') &= 2\delta(x-x'), \\
(2\ell+1)P_\ell(x) &= \frac{d}{dx}\left[P_{\ell+1}(x)-P_{\ell-1}(x)\right], \\
(\ell+1)P_{\ell+1}(x) &= (2\ell+1)xP_\ell(x)-\ell P_{\ell-1}(x), \\
P_\ell(x) &= \frac{1}{2^\ell \ell!}\frac{d^\ell}{dx^\ell}(x^2-1)^\ell \;\; \text{(Rodriguez formula)}.
\end{aligned}$$

(b) Find a_ℓ for all ℓ for the potential

$$\Phi(r=R,\cos\theta) = V_0\cos(2\theta).$$

Assuming the inside of the sphere is empty, write the potential $\Phi(\vec{r})$ for all $\vec{r}$.

12. Like the previous problem, but with the potential

$$\Phi(r=R,\cos\theta) = \begin{cases} V_0, & \cos\theta > 0 \\ -V_0 & \cos\theta < 0 \end{cases}$$

(a) Using the identities from the previous problem, show that for this potential

$$a_\ell = V_0 P_{\ell-1}(x=0)\frac{(2\ell+1)}{(\ell+1)}.$$

(b) Again, using the identities above, show that

$$\begin{aligned}
P_{\ell+1}(x=0) &= -\frac{\ell}{(\ell+1)}P_{\ell-1}(x=0), \\
P_{\ell-1}(x=0) &= -\frac{(\ell-2)}{(\ell-1)}P_{\ell-3}(x=0).
\end{aligned}$$

(c) Putting these together, show that

$$\begin{aligned}
a_\ell &= -a_{\ell-2}\frac{(2\ell+1)(\ell-2)}{(\ell+1)(2\ell-3)}, \\
a_1 &= 3V_0/2, \quad a_{(\text{even})} = 0.
\end{aligned}$$

(d) To test your answer, write a short program to calculate $\Phi(r=R)$ and see whether it matches the expectation.

5 Multipole Expansions

Here, we consider fields due to compact figurations of static charges when viewed from far away. The fields are dominated by the lowest non-zero moment of the charge distributions, e.g. monopole, dipole, quadrupole, etc.

5.1 Expanding Coulomb Potential in Spherical Harmonics

Consider a point charge Q at position $a\hat{z}$. We wish to express the Coulomb potential as an expansion in $1/r$, where if $a = 0$ only the first term survives.

$$\begin{aligned} \frac{Q}{|\vec{r} - a\hat{z}|} &= \frac{Q}{\sqrt{r^2 - 2ar\cos\theta + a^2}} \\ &= \frac{Q}{r}\frac{1}{\sqrt{1 - 2(a/r)\cos\theta + (a/r)^2}}. \end{aligned} \tag{5.1}$$

We now treat $\epsilon = 2(a/r)\cos\theta - (a/r)^2$ as a small number

$$\begin{aligned} \frac{Q}{|\vec{r} - a\hat{z}|} &= \frac{Q}{r}\frac{1}{\sqrt{1-\epsilon}} \\ &= \frac{Q}{r}\left(1 + \epsilon/2 + 3\epsilon^2/8 + 15\epsilon^3/48 \cdots\right). \end{aligned} \tag{5.2}$$

Organizing terms by powers of a/r,

$$\begin{aligned} \frac{Q}{|\vec{r} - a\hat{z}|} &= \frac{Q}{r}\left\{1 + (a/r)\cos\theta + (a/r)^2\frac{1}{2}(3\cos^2\theta - 1)\right. \\ &\qquad \left. +(a/r)^3\frac{1}{2}(5\cos^3\theta - 3\cos\theta) + \cdots\right\}. \end{aligned} \tag{5.3}$$

One can see that this matches up with the expansion for the Legendre polynomials,

$$\frac{Q}{|\vec{r} - a\hat{z}|} = \frac{Q}{r}\sum_\ell \left(\frac{a}{r}\right)^\ell P_\ell(\cos\theta). \tag{5.4}$$

The above expression assumed the charge was along the z axis. One could prove this expression term-by-term, or simply apply the generating function for Legendre polynomials (copied from Eq. (4.34)),

$$\frac{1}{\sqrt{1 - 2xt + t^2}} = \sum_\ell P_\ell(x)t^\ell. \tag{5.5}$$

From one perspective, this generating function defines the Legendre polynomials, and other properties such as recurrence relations derive from the generating function.

For our multipole expansion, we wish to find an expression for the field at some angle θ, ϕ when the charge is at angle θ', ϕ'. The addition theorem for spherical harmonics,

$$P_\ell(\cos\gamma) = \frac{4\pi}{2\ell+1}\sum_{m=-\ell}^{\ell} Y^*_{\ell m}(\theta',\phi')Y_{\ell,m}(\theta,\phi), \tag{5.6}$$

plays a critical role in going forward with the multipole expansion. The angle γ is the angle between the diretions θ, ϕ and θ', ϕ'. Here is the proof, along with explanations of each step,

$$\begin{aligned}
&(1) & P_\ell(\cos\gamma) &= P_\ell(\cos\gamma)P_\ell(1) & (5.7)\\
&(2) & &= \frac{4\pi}{(2\ell+1)}Y^*_{\ell 0}(0,0)Y_{\ell 0}(\gamma,0)\\
&(3) & &= \frac{4\pi}{(2\ell+1)}\langle\theta=0,\phi=0|\ell,0\rangle\langle\ell,0|\gamma,0\rangle\\
&(4) & &= \frac{4\pi}{(2\ell+1)}\sum_m\langle\theta=0,\phi=0|\ell,m\rangle\langle\ell,m|\gamma,0\rangle\\
&(5) & &= \frac{4\pi}{(2\ell+1)}\sum_m\langle\theta=0,\phi=0|\mathcal{R}^{-1}\mathcal{R}|\ell,m\rangle\langle\ell,m|\mathcal{R}^{-1}\mathcal{R}|\gamma,0\rangle\\
&(6) & &= \frac{4\pi}{(2\ell+1)}\sum_m\langle\theta=0,\phi=0|\mathcal{R}^{-1}|\ell,m\rangle\langle\ell,m|\mathcal{R}|\gamma,0\rangle\\
&(7) & &= \frac{4\pi}{(2\ell+1)}\sum_m\langle\theta',\phi'|\ell,m\rangle\langle\ell,m|\phi,\theta\rangle\\
&(8) & &= \frac{4\pi}{(2\ell+1)}\sum_m Y^*_{\ell m}(\theta',\phi')Y_{\ell m}(\theta,\phi). & (5.8)
\end{aligned}$$

1. Use the fact that $P_\ell(\cos\theta = 1) = 1$.
2. Use the fact that $P_\ell(\cos\theta) = \sqrt{4\pi/(2\ell+1)}Y_{\ell 0}(\theta,\phi)$.
3. Writing the $Y_{\ell m}$ in bra-ket notation so that completeness is more apparent.
4. $Y_{\ell m}(\theta = 0) = 0$ for all $m \neq 0$, so extra parts of sum can be added without changing result.
5. R are rotations that move $\gamma, 0$ to θ, ϕ and $0, 0$ to θ', ϕ'.
6. Rotations don't affect completeness $|m\rangle\langle m|$.
7. Using definitions of rotation, note γ is angle between θ', ϕ' and θ, ϕ.
8. Leaving bra-ket notation.

The potential depends on the relative angle γ, so we can rewrite it as

$$\frac{Q}{|\vec{r}-\vec{a}|} = \frac{4\pi Q}{r}\sum_{\ell m}\frac{1}{(2\ell+1)}\left(\frac{a}{r}\right)^\ell Y^*_{\ell m}(\theta',\phi')Y_{\ell m}(\theta,\phi), \tag{5.9}$$

where the potential is evaluated at r, θ, ϕ due to a charge at a, θ', ϕ'.

Equation (5.9) has profound implications due to the fact that the same ℓm combination appears in both spherical harmonics. If one expresses the angular distribution of the potential at some distance r outside the charge distribution, the moments ℓ, m will depend only the moments of the local charge distribution with the same ℓ and m. To see this, we consider an array of charges, represented by a charge density $\rho(\vec{r})$. Eq. (5.9) then gives

$$\Phi(\vec{r}) = \frac{4\pi}{r}\int d^3r'\,\rho(\vec{r}')\sum_{\ell m}\frac{1}{(2\ell+1)}\left(\frac{r'}{r}\right)^{\ell} Y^*_{\ell m}(\theta',\phi')Y_{\ell m}(\theta,\phi), \tag{5.10}$$

where θ, ϕ describe the direction of $\vec{r}$ and θ', ϕ' describe the direction of r'. For a fixed magnitude r, the θ, ϕ dependence of $\Phi(\vec{r})$ can be described by coefficients, $\Phi_{\ell m}(r)$. Using the completeness relation for spherical harmonics,

$$\int d\Omega\, Y_{\ell' m'}(\Omega)Y_{\ell m}(\Omega) = \delta_{\ell\ell'}\delta_{mm'}, \tag{5.11}$$

one can see that the coefficients $\Phi_{\ell m}$ can be consistently defined as

$$\begin{aligned}\Phi(r,\theta,\phi) &= \sum_{\ell m}\frac{1}{(2\ell+1)}\Phi_{\ell m}(r)Y_{\ell m}(\theta,\phi), \\ \Phi_{\ell m}(r) &= (2\ell+1)\int d\Omega\, Y^*_{\ell,m}(\theta,\phi)\Phi(r,\theta,\phi).\end{aligned} \tag{5.12}$$

Thus, Eq. (5.9) becomes

$$\begin{aligned}\Phi_{\ell m}(r) &= \int d\Omega\, Y^*_{\ell,m}(\theta,\phi)\frac{4\pi}{r}\int d^3r'\,\rho(\vec{r}')\sum_{\ell' m'}\left(\frac{r'}{r}\right)^{\ell'} Y^*_{\ell' m'}(\theta',\phi')Y_{\ell' m'}(\theta,\phi) \\ &= \frac{4\pi}{r^{\ell+1}}q_{\ell m},\end{aligned} \tag{5.13}$$

where $q_{\ell m}$ are the multipole moments of the charge distribution,

$$q_{\ell m} = \int d^3r'\, r'^{\ell}\rho(\vec{r}')Y^*_{\ell m}(\theta',\phi'). \tag{5.14}$$

The potential can then be written as a sum over harmonic with coefficients given by $q_{\ell m}$,

$$\Phi(\vec{r}) = \sum_{\ell m}\frac{4\pi}{(2\ell+1)}q_{\ell m}\frac{Y_{\ell m}(\theta,\phi)}{r^{\ell+1}}. \tag{5.15}$$

The moments, $q_{\ell m}$ have dimension charge multiplied by length to the ℓ^{th} power, and the effect of higher moments falls off as $1/r^{\ell+1}$.

The lowest multipole moments can be re-expressed using the forms for the spherical harmonics in Eq.s (4.31),

$$\begin{aligned}
q_{00} &= \frac{1}{\sqrt{4\pi}}\int d^3r\,\rho(\vec{r}) = \frac{1}{\sqrt{4\pi}}q, \\
q_{11} &= -\sqrt{\frac{3}{8\pi}}\int d^3r\,\rho(\vec{r})(x-iy) = -\sqrt{\frac{3}{8\pi}}(p_x - ip_y), \\
q_{10} &= -\sqrt{\frac{3}{4\pi}}\int d^3r\,\rho(\vec{r})z = -\sqrt{\frac{3}{4\pi}}p_z, \\
q_{22} &= \sqrt{\frac{15}{32\pi}}\int d^3r\,\rho(\vec{r})(x-iy)^2 = \sqrt{\frac{15}{288\pi}}(Q_{11} - 2iQ_{12} - Q_{22}), \\
q_{21} &= -\sqrt{\frac{15}{8\pi}}\int d^3r\,\rho(\vec{r})(x-iy)z = -\sqrt{\frac{15}{72\pi}}(Q_{13} - iQ_{23}), \\
q_{20} &= \sqrt{\frac{5}{16\pi}}\int d^3r\,\rho(\vec{r})(3z^2 - r^2) = \sqrt{\frac{5}{16\pi}}Q_{33}.
\end{aligned} \tag{5.16}$$

Here q is the net charge,

$$q = \int d^3r\,\rho(\vec{r}), \tag{5.17}$$

p_i are the dipole moments,

$$p_i = \int d^3r\,\rho(\vec{r})r_i, \tag{5.18}$$

and Q_{ij} are the quadrupole moments,

$$Q_{ij} = \int d^3r\,\rho(\vec{r})(3r_ir_j - r^2\delta_{ij}). \tag{5.19}$$

Example 5.1:
Consider the charge density

$$\rho(\vec{r}) = Axe^{-(x^2+y^2+z^2)/2R^2}.$$

Find all the multipole moments, $q_{\ell m}$ for $\ell \leq 2$.
Solution:
First find the net charge q, the dipole moments p_i and the quadrupole tensor Q_{ij}. The net charge $q = 0$ because the charge density is odd in x. Again, by symmetry, the only non-zero dipole moment is p_x,

$$\begin{aligned}
p_x &= A\int d^3r\,x^2 e^{-(x^2+y^2+z^2)/2R^2} \\
&= (2\pi R^2)^{3/2}AR^2.
\end{aligned}$$

The quadrupole tensor elements, Q_{ij}, are all zero because they all involve an overall even power of coordinates. Some involve odd powers of x which would permit a non-zero result for the integral over x, but they then involve odd powers of y or z, which would then be zero. From Eq.s (5.16), the only non-zero multipole moments are q_{11} and q_{1-1},

$$\begin{aligned} q_{11} &= -\sqrt{3}\pi A R^{7/2}, \\ q_{1-1} &= -q_{11}^* = \sqrt{3}\pi A R^{7/2}. \end{aligned}$$

The sign in the last term was found by using Eq. (4.33).

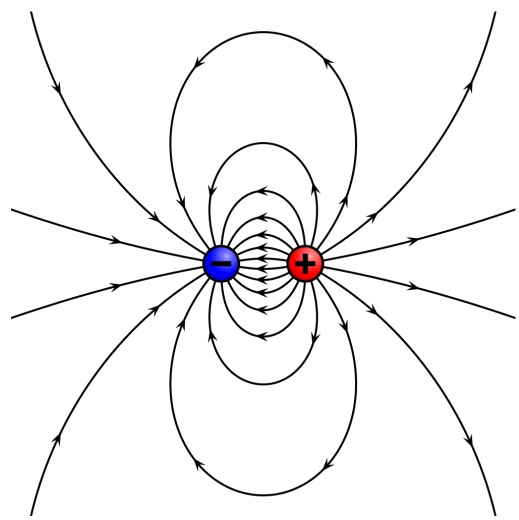
5.2 Electric Field of a Dipole

Here we consider a dipole moment $p_z = p\hat{z}$, due to a small charge distribution. By small, we assume that r is much larger than any separation of charges. Only the $\ell = 1, m = 0$ multipole is not zero. From Eq. (5.16)

$$q_{10} = p\sqrt{\frac{3}{4\pi}}. \tag{5.20}$$

The potential for a dipole, from Eq. (5.15) is

$$\begin{aligned} \Phi(\vec{r}) &= p\sqrt{\frac{3}{4\pi}}\frac{4\pi}{3r^2}Y_{10}(\theta,\phi) \qquad (5.21) \\ &= \frac{p\cos\theta}{r^2} = p\frac{z}{r^3}. \end{aligned}$$



Electric field lines as described with the $\hat{z}$ pointing to the right.

The electric field for various components is

$$\begin{aligned} E_z &= -\frac{p}{r^3} + \frac{3pz^2}{r^5}, \\ E_x &= \frac{3pxz}{r^5}, \quad E_y = \frac{3pyz}{r^5}, \\ E_r &= \frac{2p\cos\theta}{r^3}, \\ E_\theta &= \frac{p\sin\theta}{r^3}, \\ \vec{E} &= -\frac{1}{r^3}\vec{p} + 3\frac{\vec{p}\cdot\vec{r}}{r^5}\vec{r}. \end{aligned}$$

In a constant electric field, $\Phi = -Ez$, the interaction energy of the field with the dipole is

$$\begin{aligned} U &= \int d^3r\, \rho(\vec{r})(-Ez) \qquad (5.22) \\ &= -p_z|\vec{E}| = -\vec{p}\cdot\vec{E}. \qquad (5.23) \end{aligned}$$

For two dipoles, with dipole moments $\vec{p}_a$ and $\vec{p}_b$, the interaction energy between the two can be found by considering the interaction energy of dipole b due to the field generated by dipole a,

$$\begin{aligned} U &= -\vec{p}_b \cdot \vec{E}_a, \\ \vec{E}_b &= -\frac{1}{r^3}\vec{p}_b + \frac{(\vec{p}_b \cdot \vec{r})}{r^5}\vec{r}, \;\; \vec{r} \equiv \vec{r}_a - \vec{r}_b, \\ U &= \frac{\vec{p}_a \cdot \vec{p}_b}{r^3} - \frac{(\vec{p}_a \cdot \vec{r})(\vec{p}_b \cdot \vec{r})}{r^5}. \end{aligned} \tag{5.24}$$

In an extremely hot system, the average energy between two dipoles vanishes because on average the directions of the dipoles are random and unaligned. However, if the system is at a finite temperature T, a dipole moment is induced because the energy is lower if the dipole points parallel to the field. Finding the value of the average induced moment depends on how many orientation states exist for the dipole, which is a matter for quantum mechanics. Classically, one would average over all directions of the dipole,

$$\begin{aligned} \langle p_z \rangle &= \frac{\int_{-1}^{1} d\cos\theta \; p\cos\theta e^{pE\cos\theta/T}}{\int_{-1}^{1} d\cos\theta \; e^{pE\cos\theta/T}} \\ &= \frac{p}{\tanh(pE/T)} - \frac{T}{E}. \end{aligned} \tag{5.25}$$

For high temperatures, one can expand in pE/T and find

$$\langle p_z \rangle = \frac{Ep^2}{3T}.$$

The average interaction energy with an electric field is then

$$\langle U \rangle = -\langle \vec{p} \cdot \vec{E} \rangle = -\frac{E^2p^2}{3T}, \tag{5.26}$$

which is attractive. If the dipole in question, #1, is due to another dipole, #2, its interaction energy depends only $\langle E^2 \rangle$ due to the dipole #2. If the two dipoles are separated by a distance r, and if the second dipole is basically randomly oriented (again, consistent with the high T limit), the average squared field is

$$\begin{aligned} \langle E^2 \rangle &= \frac{1}{2}\int_{-1}^{1} d\cos\theta \; \left[E_r^2(r,\theta) + E_\theta^2(r,\theta)\right] \\ &= \frac{p^2}{r^6}\int_0^1 d\cos\theta \; (4\cos^2\theta + \sin^2\theta) \\ &= \frac{2p^2}{r^6}. \end{aligned} \tag{5.27}$$

The average interaction energy is then

$$\langle U \rangle = -\frac{4p^4}{3r^6T}. \tag{5.28}$$

An extra factor of two was added to account for the non-random part of the orientation of dipole #2 due to the random orientation with #1. This should be the dominant source of the potential between two dipoles at large distances in the limit that the temperature is high enough that the dipoles are for the most part randomly distributed. At short distance, atoms or molecules begin to repel one another once the electronic wave functions attempt to overlap one another. These forces typically have forms that die off faster than $1/r^6$, perhaps exponentially or perhaps as a higher power than 6. Indeed, the $1/r^6$ attractive force is known as the Van der Waals force and has had great phenomenological success for a variety of phenomena. A common potential is the 6-12 potential which behaves as $A/r^{12} - B/r^6$.

5.3 Energy in arbitrary external field

As stated earlier, the energy of a charge in an external potential Φ, or that of a dipole in an external electric field is

$$\begin{aligned} U_{\rm (monopole)} &= Q\Phi_0, & (5.29)\\ U_{\rm (dipole)} &= -\vec{p}\cdot\vec{E}_0. & (5.30)\end{aligned}$$

The subscript "0" emphasizes that this is an external field, not the field from the charge or the dipole itself. This can be extended to arbitrary multipoles in an external field,

$$\begin{aligned} U &= \int d^3r\ \Phi(\vec{r})\rho(\vec{r}), & (5.31)\\ \Phi(\vec{r}) &= \sum_{n_xn_yn_z} \frac{x^{n_x}y^{n_y}z^{n_z}}{n_x!n_y!n_z!}\ \partial_x^{n_x}\partial_y^{n_y}\partial_z^{n_z}\Phi\Big|_{r=0},\\ U &= \sum_{n_xn_yn_z} \frac{1}{n_x!n_y!n_z!}M_{n_xn_yn_z}\ \partial_x^{n_x}\partial_y^{n_y}\partial_z^{n_z}\Phi\Big|_{r=0},\\ M_{n_xn_yn_z} &= \int d^3r\ \rho(\vec{r})x^{n_x}y^{n_y}z^{n_z}.\end{aligned}$$

The moments of the charge distribution,

$$\begin{aligned} M_{000} &= q, & (5.32)\\ M_{100} &= p_x, M_{010} = p_y, M_{001} = p_z,\\ M_{200} &= R_{11},\ M_{110} = R_{12},\ M_{101} = R_{13},\\ M_{110} &= R_{21},\ M_{020} = R_{22},\ M_{011} = R_{23},\\ M_{101} &= R_{31},\ M_{011} = R_{32},\ M_{002} = R_{33},\\ R_{ij} &\equiv \int d^3r\ \rho(\vec{r})r_ir_j. & (5.33)\end{aligned}$$

With these definitions,

$$\begin{aligned} U &= q\Phi_0 - \vec{p}\cdot\vec{E}_0 - \frac{1}{2}R_{ij}\partial_i E_{0j}, \qquad (5.34)\\ &= q\Phi_0 - \vec{p}\cdot\vec{E}_0 - \frac{1}{6}(Q_{ij} + \bar{R}\delta_{ij})\partial_i E_{0j},\\ \bar{R} &\equiv \int d^3r\, r^2\rho(\vec{r}),\\ Q_{ij} &\equiv \int d^3r\,(3r_ir_j - r^2\delta_{ij})\rho(\vec{r}). \end{aligned}$$

where the fields and their derivatives are evaluated at $r = 0$. The term with $\bar{R}$ does not contribute to the energy because $\nabla\cdot\vec{E}_0 = 0$ (the external field is that part of the field not affected by local charge density), thus

$$U = q\Phi_0 - \vec{p}\cdot\vec{E}_0 - \frac{1}{6}Q_{ij}\partial_i E_{0j}. \qquad (5.35)$$

Here, the quadrupole moment tensor Q_{ij} is defined as it was previously in this chapter, Eq. (5.19).

5.4 Homework Problems

1. Consider four charges $+q, +q, -q, -q$ at the Cartesian positions $(a, 0, 0)$, $(0, a, 0)$, $(-a, 0, 0)$, $(0, -a, 0)$ respectively.
   (a) Find the dipole moments p_i and the quadrupole tensor Q_{ij} for all i and j.
   (b) Find all $q_{\ell m}$ for all $\ell \leq 2$.
2. Consider a charge distribution with $q_{21} = q_{2-1}$=some imaginary number iQa^2, see definitions in Eq. (5.16). Draw a figure where you place a minimum number of discrete charges that reproduces the given q_{21} and q_{2-1}, while having all other $q_{ij} = 0$ for $\ell \leq 2$.
   (a) Provide the positions and find the individual charges, all of which are $\pm q$, in terms of Q. Only place charges on a lattice where the step size is a, i.e. at positions ia, ja, ka, where i, j, k are integers.
   (b) In terms of the magnitude of the individual charges, q, and the lattice spacings a, find the potential as a function of r, θ and ϕ.
3. Consider a simple model of an atom being a particle of charge e that moves in a three-dimensional harmonic oscillator with effective spring constant k. A constant electric field E_0 is added.
   (a) What is the magnitude of the induced dipole moment p?
   (b) What is the change in total energy of the charge due to the introduction of the field? Give answer in terms of p and E_0.

4. Consider a point charge q at $\vec{r}' = a\hat{z}$.

   (a) Find the moments, $q_{\ell m}$ defined in Eq. (5.14), for all ℓ and m, defining the moments around the origin

   (b) Show that the potential calculated with $q_{\ell m}$ using Eq. (5.15) matches Eq. (5.9) for the case where the charge is along the z axis.

   (c) Show that for the case where $\Phi(\vec{r})$ is evaluated with $\vec{r}$ lying along the z-axis that the sum becomes $q/(r-a)$.

5. Any function that can be written as a sum over Cartesian polynomials of order $\leq \ell$, i.e.,

$$F(x,y,z) = \sum_{\ell_x+\ell_y+\ell_z \leq \ell} A_{\ell_x \ell_y \ell_z} x^{\ell_x} y^{\ell_y} z^{\ell_z},$$

can be expressed as a sum of spherical harmonics with order $\leq \ell$,

$$F = \sum_{\ell',m',\ell' \leq \ell} A_{\ell' m'}(r) Y_{\ell' m'}(\theta, \phi).$$

Using this fact prove that the multipole moments of order $\leq \ell$, for the case when all moments $q_{\ell' m'}$ vanish for $\ell' < \ell$, are unaffected by a translation of the origin, and that the higher moments, $> \ell$, are affected by this change of the coordinate system. This means that the dominant multipole is unaffected by a translation of the coordinate system.
Hint: Using the definition of the moments, Eq. (5.14), replace $\rho(\vec{r})$ with $\rho(\vec{r}+\vec{a})$, then express the new charge density as a Taylor expansion in powers of a.

6 Magnetostatics

Here, we consider magnetic fields from steady currents, and set the stage for fields from dynamic sources.

6.1 The Biot-Savart Law

In Chapter 2 we derived Maxwell's equations from Lagrangians written in terms of the vector potential, and for electrostatics the Maxwell equation that was $\nabla^2\Phi = -\nabla\cdot\vec{E} = -4\pi\rho$. This then led to Coulomb's law. Here, we show one can similarly relate $\nabla^2\vec{A} = 4\pi\vec{J}$, so that each component of $\vec{A}$ is related to the corresponding component in $\vec{J}$ in the same manner that $\Phi = A_0$ is related to $\rho = J_0$.

To this we begin with the Maxwell relation in a static system ($\partial_t\vec{E} = 0$),

$$\begin{aligned} \nabla\times\nabla\times\vec{A} &= 4\pi\vec{J}, \\ \nabla^2\vec{A} - \nabla(\nabla\cdot\vec{A}) &= -4\pi\vec{J}. \end{aligned} \tag{6.1}$$

This last step is a simple vector identity, $\vec{a}\times(\vec{b}\times\vec{c}) = \vec{b}(\vec{a}\cdot\vec{c}) - (\vec{a}\cdot\vec{b})\vec{c}$, with $\vec{a} = \vec{b} = \nabla$ and $\vec{c} = \vec{A}$.

Our difficulty going forward is the term $\nabla(\nabla\cdot\vec{A})$. However, one can eliminate that term by choosing a convenient gauge. As show earlier, one can add a term to the vector potential, $\vec{A}' = \vec{A} + \nabla\Lambda$, without changing any physics, because $\vec{E}$ and $\vec{B}$ are unchanged. This works for any function $\Lambda(x, y, z)$. We are in need of a new vector potential $\vec{A}'$ where $\nabla\cdot\vec{A}' = 0$,

$$\begin{aligned} \nabla\cdot\left(\vec{A} + \nabla\Lambda\right) &= 0, \\ \nabla^2\Lambda &= -\nabla\cdot\vec{A}. \end{aligned} \tag{6.2}$$

If there were some charge density, $\rho = \nabla\cdot\vec{A}$, then finding Λ would be equivalent to finding the potential corresponding to the static charge density. Of course, there is such a potential, thus there is always a gauge such that $\nabla\cdot\vec{A} = 0$. Assuming one is in that gauge,

$$\nabla^2\vec{A} = -4\pi\vec{J}. \tag{6.3}$$

Of course, due to gauge invariance there are many vector potentials one could choose, but they all lead to the same electromagnetic fields. Thus, we are free to choose this gauge.

Equation (6.3) is actually three separate equations. Each one is effectively a Poisson's equation where the components J_i play the role of charge densities. This is equivalent to Poisson's equation for the electric potential, $\nabla^2\Phi = -4\pi\rho$, so one can use the same results as before. Mainly,

$$\vec{A}(\vec{r}) = \int d^3r'\,\vec{J}(\vec{r}')\frac{1}{|\vec{r}-\vec{r}'|}. \tag{6.4}$$

The magnetic field is then

$$\begin{aligned}\vec{B}(\vec{r}) &= \nabla \times \vec{A} \\ &= \int d^3r' \, \frac{\vec{J}(\vec{r}') \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3},\end{aligned} \tag{6.5}$$

which is the Biot-Savart Law.

Note the expression assumes natural units, if the currents are in MKSA units, where Amperes (Coulombs$/s$) are the unit of current, the expression picks up an additional factor,

$$\begin{aligned}\vec{B}_{(\mathrm{MKSA})}(\vec{r}) &= \frac{\mu_0}{4\pi} \int d^3r' \, \frac{\vec{J}(\vec{r}') \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3}, \\ \mu_0 &= 4\pi 10^{-7} \ \mathrm{Tm/A}.\end{aligned} \tag{6.6}$$

Example 6.1:

A 5 Ampere current travels through a thin wire in a circular loop of radius $R = 10$ cm. Find the strength of the magnetic field in the center of the loop.

Solution:

The current density element $\vec{J} d^3r'$ becomes $\vec{I} d\ell$ for a thin wire. Each element $\vec{I} d\ell$ provides a differential component to the magnetic field that is perpendicular to the plane of the loop (we'll call it $\hat{z}$). Integrating around the loop,

$$\vec{B}_{(\mathrm{MKSA})}(\vec{r} = 0) = \frac{\mu_0 I}{4\pi} \int d\ell \, \frac{\hat{z}}{R^2}, \tag{6.7}$$

$$|\vec{B}_{(\mathrm{MKSA})}(\vec{r} = 0)| = \frac{\mu_0 I}{2R} = \pi \cdot 10^{-5} \ \mathrm{T}. \tag{6.8}$$

6.2 Magnetic Moments

If one is outside a localized current distribution, the vector potential falls off according to the same mathematics as used for multipole moments of the electric potential from the previous chapter. Because the net current density is zero for a confined static current,

$$\int d^3r \, J_i(\vec{r}) = 0, \tag{6.9}$$

the "monopole" charge for each component i vanishes. Whereas $\vec{p}$ referred to the dipole moment of a charge distribution, here we will worry about equivalent of a dipole moment vector for each component J_i. Thus, we consider the moment

$$\pi_{ij} = \int d^3r' \, J_i(\vec{r}') r_j'. \tag{6.10}$$

Far away, the dipole contribution to the $i^{\rm th}$ component of the vector potential is

$$A_i(\vec{r}) = \frac{\pi_{ij} r_j}{r^3}, \tag{6.11}$$

where we have simply translated from the result for the electric potential for a dipole, $\Phi = \vec{p} \cdot \vec{r}/r^3$. Next, one can find the magnetic field,

$$\begin{aligned} B_i &= \epsilon_{ijk} \partial_j A_k \\ &= \epsilon_{ijk} \left(\frac{\pi_{kj}}{r^3} - 3 \frac{\pi_{k\ell} r_\ell r_j}{r^5} \right). \end{aligned} \tag{6.12}$$

Using the identity,

$$\epsilon_{ijk} \epsilon_{k\ell n} = \delta_{i\ell} \delta_{jn} - \delta_{in} \delta_{j\ell}, \tag{6.13}$$

One can see that

$$\begin{aligned} \frac{1}{2} \epsilon_{ijk} \epsilon_{k\ell n} \pi_{\ell n} &= \frac{1}{2} (\pi_{ij} - \pi_{ji}) \\ &= \pi_{ij}, \end{aligned} \tag{6.14}$$

where the last step used the fact that π_{ij} is anti-symmetric (which follows from current conservation, see end-of-chapter problem). Defining the magnetic moment $\vec{m}$,

$$\begin{aligned} m_i &\equiv -\frac{1}{2} \epsilon_{ijk} \pi_{jk} = \frac{1}{2} \epsilon_{ijk} \pi_{kj}, \\ \vec{m} &= \frac{1}{2} \int d^3r \, \vec{r} \times \vec{J}. \end{aligned} \tag{6.15}$$

one can express π_{ij} in terms of $\vec{m}$,

$$\pi_{ij} = -\epsilon_{ijk} m_k. \tag{6.16}$$

Inserting these into the expression for $\vec{B}$ in Eq. (6.12),

$$\begin{aligned} B_i &= \frac{2m_i}{r^3} + \frac{3}{r^5} \epsilon_{ijk} \epsilon_{k\ell n} m_n r_j r_\ell \\ &= \frac{2m_i}{r^3} + \frac{3}{r^5} \left(\delta_{i\ell} \delta_{jn} - \delta_{in} \delta_{\ell j} \right) m_n r_j r_\ell \\ &= -\frac{m_i}{r^3} + \frac{3r_i}{r^5} (\vec{m} \cdot \vec{r}) \\ \vec{B} &= -\frac{\vec{m}}{r^3} + \frac{3\vec{r}}{r^5} (\vec{m} \cdot \vec{r}). \end{aligned} \tag{6.17}$$

For a current loop with a thin wire, the magnetic moments have a simple geometric expression. For a thin wire one can

$$\begin{aligned} d^3r &= d\ell d^2r_\perp, \\ d\ell \int d^2r_\perp \vec{J}(\vec{r}) &= \vec{I} d\ell, \\ \vec{m} &= \frac{1}{2} \int d\ell \, \vec{r} \times \vec{I} \\ &= \frac{I}{2} \int \vec{r} \times d\vec{\ell}. \end{aligned} \tag{6.18}$$

If the current lies in a plane, $|\vec{m}| = Ia$, where a is the area of the loop, and the direction of $\vec{m}$ is perpendicular to the plane. Here, $d\ell$ is along the direction of the wire and $\vec{r}_\perp$ is perpendicular to the wire.

6.3 Magnetic Forces and Torques

The magnetic force $q\vec{v} \times \vec{B}$ leads to torques on small current loops. In a magnetic field $\vec{B}$, the net force acting on some set of currents $\vec{J}$ is

$$\vec{F} = \int d^3r\, \vec{J} \times \vec{B}, \tag{6.19}$$

because $\vec{J}d^3r$ is the differential contribution to $q\vec{v}$. For a constant magnetic field $\vec{B}$ factors out from the integral and one can see that the net force acting on a static current distribution is zero because $\int d^3r\, \vec{J} = 0$ (See end-of-chapter problem). Nonetheless, there is a net torque,

$$\begin{aligned} \vec{\tau} &= \int d^3r'\, \vec{r} \times (\vec{J}(\vec{r}') \times \vec{B}), \\ \tau_i &= B_k \int d^3r'\, r'_k J_i(\vec{r}') - B_i \int d^3r'\, r'_k J_k(\vec{r}'). \end{aligned} \tag{6.20}$$

Now one can use that fact that

$$\pi_{ik} \equiv \int d^3r'\, r'_i B_k(\vec{r}') \tag{6.21}$$

is anti-symmetric to throw away the second term, and rewrite the first term,

$$\begin{aligned} \tau_i &= \frac{1}{2} B_k \int d^3r' \left[r'_k J_i(\vec{r}') - r'_i J_k(\vec{r}') \right] \\ \vec{\tau} &= \frac{1}{2} \left[\int d^3r'\, \vec{r}' \times \vec{J}(\vec{r}') \right] \times \vec{B} \\ &= \vec{m} \times \vec{B}. \end{aligned} \tag{6.22}$$

The work done by the torque in rotating an object is $\tau d\theta$, so the potential energy of the dipole is

$$U = -\vec{m} \cdot \vec{B}, \tag{6.23}$$

and magnetic dipoles prefer to be aligned parallel to a magnetic field.

For two physically separated dipoles, $\vec{m}_a$ and $\vec{m}_b$, one can then calculate the energy associated with their spin configuration using Eq. (6.17) which gives the magnetic field due to dipole a that would be experienced by dipole b,

$$\begin{aligned} U &= -\vec{m}_b \cdot \left[-\frac{\vec{m}_a}{r^3} + \frac{3\vec{r}}{r^5} (\vec{m}_a \cdot \vec{r}) \right] \\ &= \frac{\vec{m}_a \cdot \vec{m}_b}{r^3} - \frac{3\vec{r}}{r^5} (\vec{m}_a \cdot \vec{r})(\vec{m}_b \cdot \vec{r}). \end{aligned} \tag{6.24}$$

6.4 Overlapping Distributions and Hyperfine Splitting

Thus far, we have considered only the interaction between to distributions separated by a finite amount. Sometimes, e.g. electrons interacting with a nucleus to produce the hyper-fine interaction, the two distributions lay atop one another. Here, we consider the electric and magnetic field integrated over volumes that include all charges. This does not provide the entire distribution of the field, but for energies such as $-\vec{p}\cdot\vec{E}$ or $-\vec{m}\cdot\vec{B}$, it will give the interaction energies for a smooth distribution of electric or magnetic dipole densities within a volume. This mainly comes into play in the interior of atoms. For examples, the smooth distribution of electron density, and therefore the magnetic-dipole density, are well known from solving the Schrödinger equation. This interacts with the magnetic field due to the nuclear magnetic dipole. Because the electron density is smooth, one needs only integrate the magnetic dipole density of the electrons multiplied by the magnetic field due to the nuclear magnetic dipole.

To calculate the magnetic interaction of a nuclear magnetic moment, $\vec{\mu}_N$, with the intrinsic spin of electrons electrons in an atom, ρ_e, one would do the integral

$$U = -\mu_e \int d^3r\ [\rho_{e,\uparrow}\ (\vec{r}) - \rho_{e,\downarrow}(\vec{r})]\,\vec{B}_z(\vec{r}), \tag{6.25}$$

where B_z would be the field due to the nuclear magnetic moment. The magnetic moment of an electron is

$$\mu_e = g_e\frac{e\hbar}{2m_e}, \tag{6.26}$$

where $g_e \approx 2.0$ is the $g-$factor for electrons, and the $\uparrow$ and $\downarrow$ symbols denote the densities of spin-up or spin-down electrons respectively. The difficulty with performing the integral in Eq. (6.25) is that we know how to calculate the magnetic field due to a magnetic moment at distances outside the nucleus, but within the current it is more difficulty. This is a small volume, but the fields are strong. Also due to the smallness, the density of electrons can be taken as constant within it, which means we need only the integrated value of B_z. The energy within this small spherical volume of radius we denote as

$$U_\Omega \approx -\mu_e(\rho_{e,\uparrow}\ - \rho_{e,\downarrow})(r=0)\int_{r<R} d^3r\ B_z(\vec{r}). \tag{6.27}$$

This approximation is excellent because the radius of a nucleus is $\sim 10^{-4}$ that of the electron cloud. Our goal is then to calculate

$$\vec{B}_V = \int_{r<R} d^3r\ \vec{B}(\vec{r}), \tag{6.28}$$

due to the nuclear magnetic moment

$$\vec{\mu}_N = g_N\frac{e\hbar}{2M_N}\hat{z}. \tag{6.29}$$

We will consider both the cases of the electric and magnetic fields. First, we find an expression for $\vec{E}_V$ and $\vec{B}_V$ for the case where all charges and currents are within the volume, then move to the case where all charges and currents are within the volume. The electric potential in a volume with no charges can be expanded in spherical harmonics,

$$\Phi(\vec{r}) = \Phi(\vec{r}=0) - \vec{E}(r=0)\cdot\vec{r} + \sum_{\ell>2,m} A_{\ell m} Y_{\ell m}(\theta,\phi) r^\ell. \tag{6.30}$$

Calculating the integrated electric field, only the $\ell = 1$ term contributes and one finds,

$$\begin{aligned} \vec{E}_V &= -\int_{r<R} d^3r\ \nabla\Phi \\ &= \frac{4\pi R^3}{3}\vec{E}(r=0), \end{aligned} \tag{6.31}$$

because all other terms will be constructed from $Y_{\ell,m}$s with $\ell > 0$ so they vanish. For example, if one applies the gradient operator, which transforms as an $\ell = 1$ object, to $r^\ell Y_{\ell,m}$, the resulting pieces will consist of terms that behave as $Y_{\ell-1,m'}$ and $Y_{\ell+1,m}$, which will all integrate to zero for $\ell \geq 2$. Thus, if there are no charges within a spherical volume, the integrated electric field is the the electric field at the center multiplied by the volume. This means that the average electric field within a spherical volume with no charges is the electric field at the center.

Similarly, one can do the same for the magnetic field. Again, one can make an expansion for the vector potential components in the Coulomb gauge, $\vec{A}$, in spherical harmonics because $\nabla^2\vec{A} = 0$ in a region with no currents,

$$\vec{A}(\vec{r}) = \vec{A}(\vec{r}=0) - \frac{1}{2}\vec{r}\times\vec{B}(r=0) + \sum_{\ell>2,m} A_{\ell m} Y_{\ell m}(\theta,\phi) r^\ell. \tag{6.32}$$

When calculating

$$\begin{aligned} \vec{B}_V &= \int_{r<R} d^3r\ \nabla\times\vec{A} \\ &= \frac{4\pi R^3}{3}\vec{B}(r=0). \end{aligned} \tag{6.33}$$

Thus the average magnetic field in a volume with no currents is the magnetic field at the center.

Next, we calcuate the average field in a spherical volume that contains all the charges or currents. First, we consider the electric field

$$\begin{aligned} \vec{E}_V &= \int_{r<R} d^3r\ \vec{E}(\vec{r}) \\ &= -\int d^3r\ \nabla\Phi \\ &= -R^2\int d\Omega\ \hat{r}\Phi(R,\Omega). \end{aligned} \tag{6.34}$$

All the charges are inside R, so one can expand Φ in multipoles. Using the fact that the unit vector in the outward direction is

$$\hat{r} = \sin\theta\cos\phi\hat{x} + \sin\theta\sin\phi\hat{y} + \cos\theta\hat{z}, \tag{6.35}$$

one can see that $\hat{r}$ only has $\ell = 1$ terms of Φ contribute when the angular dependencies are expressed in $Y_{\ell m}$s. Thus, only the $\ell = 1$ parts of Φ contribute, which are the dipole pieces.

$$\begin{aligned}
\vec{E}_V &= -R^2 \int d\Omega \, \frac{\vec{p}\cdot\vec{r}}{r^3}\hat{r}, \\
E_{V,i} &= -R^2 \int d\Omega \, \frac{p_j r_j}{r^3}\hat{r}_i, \\
\vec{E}_V &= -\frac{4\pi}{3}\vec{p}.
\end{aligned} \tag{6.36}$$

To convince yourself the last step is correct, define the z axis to be along $\vec{p}$, then only the $\hat{z}$ part of $\hat{r}$ contributes.

Next, one can calculate the corresponding quantity for magnetic fields. Similar as before,

$$\begin{aligned}
\vec{B}_V &= \int_{r<R} d^3r \, \vec{B}(\vec{r}) \\
&= \epsilon_{ijk} \int d^3r \, \partial_j A_k \\
&= R^2 \epsilon_{ijk} \int d\Omega \, \hat{r}_j A_k(R,\Omega).
\end{aligned} \tag{6.37}$$

One can expand A_k in the same way Φ was expanded. Again, only the $\ell = 1$ terms contribute. That term is given in Eq. (6.11),

$$\begin{aligned}
\vec{B}_V &= R^2 \epsilon_{ijk} \int d\Omega \, \hat{r}_j \frac{\pi_{k\ell} r_\ell}{r^3} \\
&= -R^2 \epsilon_{ijk}\epsilon_{k\ell n} m_n \int d\Omega \, \frac{r_j r_\ell}{r^4} \\
&= -\frac{4\pi}{3}\epsilon_{ijk}\epsilon_{kjn} m_n \\
&= \frac{8\pi}{3} m_i, \\
\vec{B}_V &= \frac{8\pi}{3}\vec{m}.
\end{aligned} \tag{6.38}$$

Summarizing these results for the electric and magnetic fields,

$$\begin{aligned}
\int_{r<R} d^3r \, \vec{E}(\vec{r}) &= \begin{cases} -4\pi\vec{p}/3, & \text{all charges within sphere} \\ \vec{E}(r=0)V, & \text{all charges outside sphere} \end{cases} \\
\vec{E}(\vec{r}) &\approx \vec{E}_{\text{multipole}}(\vec{r}) - \frac{4\pi}{3}\vec{p}\delta^3(r), \\
\int_{r<R} d^3r \, \vec{B}(\vec{r}) &= \begin{cases} 8\pi\vec{m}/3, & \text{all currents within sphere} \\ \vec{B}(r=0)V, & \text{all currents outside sphere,} \end{cases}
\end{aligned} \tag{6.39}$$

Here, V is the volume of the sphere. Putting these results together, we can express the magnetic field due to a magnetic moment as

$$\vec{B}(\vec{r}) \approx \vec{B}_{\text{multipole}}(\vec{r}) + \frac{8\pi}{3}\vec{m}\delta^3(r). \tag{6.40}$$

electron	$-2.00231930436182 \pm 0.00000000000052$
muon	$-2.0023318418 \pm 0.0000000013$
proton	$5.585694702 \pm 0.000000017$
neutron	$-3.82608545 \pm 0.00000090$

Table 1: g factors for various particles. Even the neutral neutron has a magnetic moment because it is constructed of charged up and down quarks.

These expressions assume the electric and magnetic dipoles are confined to a spatially small region, with "small" being relative to the distributions with which they interact. The construction clearly provides the correct field outside the volume of the dipole, and in the neighborhood of the dipole it integrates to the correct value. This is sufficient for the purposes of calculating the hyperfine interaction, between the magnetic field of the dipole and the smooth electron cloud.

The hyperfine interaction involves the coupling of the magnetic moments of the electron, $\vec{mu}_e$ with the magnetic fields arising from the nuclear magnetic moment, $\vec{\mu}_N$.

$$U = \frac{(\vec{\mu}_N \cdot \vec{\mu}_e)}{r^3} - \frac{3(\vec{\mu}_N \cdot \vec{r})(\vec{\mu}_e \cdot \vec{r})}{r^5} - \frac{8\pi}{3}(\vec{\mu}_N \cdot \vec{\mu}_e)\delta^3(\vec{r}) - e\frac{(\vec{\mu}_N \cdot \vec{L})}{mr^3}. \tag{6.41}$$

The last term arises because if the electron is moving with finite angular momentum, it generates a magnetic field which then couples with the nuclear magnetic moment. This term only comes into play if the net orbital angular momentum of the electrons is non-zero.

Magnetic moments have dimensions of charge multiplied by velocity. For a classical particle of charge e moving in a circle of radius R with velocity v, the current and magnetic moment would be

$$\begin{aligned} I &= \frac{e}{2\pi R/v}, \qquad (6.42) \\ \mu_{(\mathrm{class})} &= \frac{IR}{2} = \frac{ev}{2} \\ \vec{\mu} &= \frac{e\vec{L}}{2m}. \end{aligned}$$

For intrinsic spin, $\vec{S}$, this no longer holds, and we use a factor g to describe the ratio of μ to the classical expectation,

$$\vec{\mu} = g\frac{e\vec{S}}{2m} = g\frac{\vec{S}}{\hbar}\left(\frac{e\hbar}{2m}\right). \tag{6.43}$$

The last factor, $e\hbar/2m$ (or $e\hbar/2mc$) depending on the units of magnetic field) is known as a Bohr magneton. The g-factor is nearly 2.0 for point-particle fermions (e.g. electrons), differing from 2 due to higher-order perturbative corrections of order e^2. Protons are not point particles and the g-factor depends on details of the spin and angular momentum configuration of the constituent quarks.

6.5 Homework Problems

1. Prove the following using current conservation ($\nabla \cdot \vec{J} = 0$ for static systems):

   (a) Beginning with
   $$\int d^3r \, (\nabla \cdot \vec{J}(\vec{r}))r_i = 0,$$
   show that
   $$\int d^3r \, J_i(\vec{r}) = 0.$$

   (b) Beginning with
   $$\int d^3r \, (\nabla \cdot \vec{J}(\vec{r}))r_i r_j = 0,$$
   show that π_{ij} is antisymmetric, where
   $$\pi_{ij} \equiv \int d^3r \, J_i(\vec{r})r_j.$$

2. Consider two current loops, each moving in the $z = 0$ plane. Each loop has current I and radius a. The first loop is centered at $x = -a, y = 0$ and is circulating clockwise, and the second loop is centered at $x =, y = 0$ and is circulating counter-clockwise.

   (a) Calculate the analog of the quadrupole moment for calculating the $i^{\rm th}$ component of the vector potential,
   $$Q_{ik\ell} = \int d^3r \, J_i(\vec{r})(3r_k r_\ell - r^2\delta_{k\ell}).$$

   (b) Find the vector potential $A_i(\vec{r})$ far away, to order $1/r^3$. Use relations from the previous chapter, where you merely add an additional index to account for the fact you are dealing with the vector potential. You can leave your answer in terms of $Q_{ik\ell}$, but clarify which components of A_i are zero, and how each component depends explicitly on the angles θ and ϕ.

3. A particle of mass m and charge e moves in a circular orbit of radius R with angular momentum L. What is the magnitude of the magnetic field at the origin? Note this was used in writing the expression in Eq. (6.41) for the hyper-fine coupling.

4. Using Eq. (6.41), consider the hyper-fine energies for the 1s levels of a hydrogen atom. There are two levels because the electron and proton spin can couple to either zero or unity.

   (a) Which terms in Eq. (6.41) contribute to the energy?

   (b) Find the difference between the energy levels. Express your answer numerically in eV, using the electron mass and charge, and using the fact that the hydrogen atom electron density for the 1s state behaves as
   $$\rho(\vec{r}) \sim e^{-2r/a_0},$$
   $a_0 = 0.529$ Å.

(c) What would the wavelength of light be for a transition between the states.

5. Consider a parallel-plate capacitor where the area of the plates is A and the small separation is a. The charge on the plates are $\pm Q$.

   (a) What is the dipole moment of the capacitor if the plates are aligned in the z direction?

   (b) What is the electric field in the capacitor?

   (c) Determine whether Eq. (6.36) is satisfied in this case where the spherical volume engulfs the entire capacitor and one assumes the electric field is assigned to the interior of the capacitor.

7 Electromagnetic Waves

In the last several sections we have considered static systems, where we could neglect all the $\partial_t \cdots$ terms in Maxwell's equations. In this chapter we consider the propagation of waves, and wave equations clearly require the $\partial_t....$ terms.

7.1 The Wave Equation

Leaving off the current and charge density terms in Maxwell's equations,

$$\begin{aligned} \nabla \cdot \vec{E} &= 0, \\ \nabla \cdot \vec{B} &= 0, \\ (\nabla \times \vec{B}) - \partial_t \vec{E} &= 0, \\ \partial_t \vec{B} + \nabla \times \vec{E} &= 0. \end{aligned} \tag{7.1}$$

One can add the curl of the third equation to the time derivative of the fourth to obtain

$$\begin{aligned} \nabla \times (\nabla \times \vec{B}) + \partial_t^2 \vec{B} &= 0, \\ -\nabla^2 \vec{B} + \nabla(\nabla \cdot \vec{B}) + \partial_t^2 \vec{B} &= 0, \\ \nabla^2 \vec{B} &= \partial_t^2 \vec{B}, \\ \partial^2 \vec{B} &= 0. \end{aligned} \tag{7.2}$$

This represents three separate wave equations, each with a wave velocity of unity. We wrote Maxwell's equations in units where the speed of light is unity, otherwise the time derivatives would change, $\partial_t \rightarrow (1/c)\partial_t$. Similarly, one can subtract the curl of the fourth equation above from the time derivative of the second and obtain

$$\begin{aligned} \nabla^2 \vec{E} &= \partial_t^2 \vec{E}, \\ \partial^2 \vec{E} &= 0. \end{aligned} \tag{7.3}$$

Equations (7.2) and (7.3) appear to represent six wave equations. For propagation along in the direction of a wavenumber $\vec{k}$ axis, the solutions have the forms

$$\begin{aligned} E_i(\vec{r}, t) &= a_i e^{i\vec{k}\cdot\vec{r} - i\omega t}, \\ B_i(\vec{r}, t) &= b_i e^{i\vec{k}\cdot\vec{r} - i\omega t}, \\ \omega &= |\vec{k}|. \end{aligned} \tag{7.4}$$

To discern the direction of propagation, one can ask for what $\delta\vec{r}$ would one considers a wave packet with combinations of different wave numbers centered around $\boldsymbol{k}$. The peak of the packet is defined by that point in coordinate space where the various wave numbers contribute with a steady phase, i.e.

$$\begin{aligned}
\partial_{k_i}\left(i\vec{k}\cdot\vec{r} - i\omega(\vec{k})t\right) &= 0, \\
\vec{r}_i - \frac{\partial\omega(\vec{k})}{\partial k_i}t &= 0, \\
\vec{r}_i &= \frac{\vec{k}}{|\vec{k}|}t.
\end{aligned} \tag{7.5}$$

The last step made use of the fact that $\omega = |\vec{k}|$. If one were in a medium, the group velocity, $d\omega/d\vec{k}$, could differ from the speed of light, a subject for the next course.

The six solutions in Eq. (7.4) are not independent. In order to satisfy the connection between $\vec{E}$ and $\vec{B}$ in Maxwell's equations in Eq.s (7.1), once given $\vec{E}$, one can find $\vec{B}$, and vice versa. Also, in order to satisfy $\nabla\cdot\vec{E} = 0$ and $\nabla\cdot\vec{B} = 0$, the wave equations must satisfy

$$\begin{aligned}
\vec{a}\cdot\vec{k} &= 0, \\
\vec{b}\cdot\vec{k} &= 0.
\end{aligned} \tag{7.6}$$

Thus, the six solutions for a given wave number become two. The amplitudes $\vec{a}$ and $\vec{b}$ must be normal to the wave number $\vec{k}$ and additionally, the third and fourth Maxwell's equations require that

$$\begin{aligned}
\vec{k}\times\vec{a} &= \omega\vec{b}, \\
\vec{k}\times\vec{b} &= -\omega\vec{a}.
\end{aligned} \tag{7.7}$$

There are two independent components of $\vec{a}$ given $\vec{k}$. This defines the polarization. Once given $\vec{a}$, $\vec{b}$ is also determined. The vector $\vec{b}$ will be normal to both $\vec{a}$ and $\vec{k}$. A solution with $a_x \neq 0$ and $a_y = 0$ is considered linearly polarized in the x direction, whereas the solutions with $a_x = 0$ is linearly polarized in the y direction. One can consider $a_y = ia_x$ and find a solution where the real part of E_x is a cosine wave and the real part of E_y is a sine wave. These are circularly polarized solutions.

7.2 Stress-Energy Tensor of Electromagnetic Waves

The zero-zero component of the stress-energy tensor is the energy density. Using the solutions for the waves in Eq. (7.4) and the stress-energy tensor for the electromagnetic field from Eq.

(3.33),

$$\begin{aligned}
T_{00} &= \frac{1}{8\pi}\left(|\vec{E}|^2+|\vec{B}|^2\right) &(7.8)\\
&= \frac{a_i^2+b_i^2}{8\pi}=\frac{|\vec{a}|^2}{4\pi}\cos^2(\vec{k}\cdot\vec{r}-\omega t),\\
T_{0i} &= \epsilon_{ijk}\frac{E_jB_k}{4\pi}\\
&= \hat{k}_i\frac{|\vec{a}|^2}{4\pi}\cos^2(\vec{k}\cdot\vec{r}-\omega t),\\
T^{ij}=-T^i{}_j &= \frac{1}{8\pi}(\delta_{ij}(E^2+B^2)-2E_iE_j-2B_iB_j)\\
&= \frac{1}{4\pi}\left\{|\vec{a}|^2\delta_{ij}-a_ia_j-b_ib_j\right\}\cos^2(\vec{k}\cdot\vec{r}-\omega t).
\end{aligned}$$

The factors $\cos^2(\vec{k}\cdot\vec{r}-\omega t)$ arise from using only the real part of $e^{i(\vec{k}\cdot\vec{r}-\omega t)}$ and assuming $\vec{a}$ is real (equivalent to saying the polarization is linear). Otherwise, phases, e.g. $\cos(\vec{k}\cdot\vec{r}-\omega t+\phi)$ would enter.

For a wave moving along the z axis, with polarization components a_x and a_y,

$$\begin{aligned}
T^{00} &= \frac{a_x^2+a_y^2}{4\pi}\cos^2(kz-\omega t), &(7.9)\\
T^{0z} &= \frac{a_x^2+a_y^2}{4\pi}\cos^2(kz-\omega t)=T^{00},\\
T^{zz} &= \frac{a_x^2+a_y^2}{4\pi}\cos^2(kz-\omega t)=T^{00},\\
T^{xx} &= T^{yy}=T^{i\neq j}=0.
\end{aligned}$$

For the last expression, we used the fact that $b_x=-a_y$ and $b_y=a_x$, which comes from Eq. (7.7). These results are expected because T_{0z} is both the momentum density and the flux of energy. The energy flux is energy density multiplied by velocity, so one expects it to equal T_{00} because the velocity in unity. Further, T_{zz} is the flux of momentum, momentum density multiplied by velocity, so it equal to $T_{0z}=T_{00}$.

7.3 The Doppler Effect

An observer sees light from a source with wave number $\vec{k}$. Its frequency is $|\vec{k}|$. If the source is moving with velocity $\vec{v}$, one can find the frequency of the emission according to an observer moving with the source, ω_s, by treating the frequency and wave number as as four vector,

$$k = (\omega,\vec{k}). \tag{7.10}$$

The frequency in the source frame is then found by the Lorentz transformation,

$$\omega_s = \gamma(\omega-\vec{v}\cdot\vec{k})=\omega\gamma(1-v\cos\theta), \tag{7.11}$$

where θ is the angle between $\vec{v}$ and $\vec{k}$. If the source is moving directly toward the observer,

$$\omega_s = \omega\sqrt{\frac{1-v}{1+v}}. \tag{7.12}$$

If the source is moving away from the observer, the sign of v shifts, and the observed frequency is less than the source frequency (red shift). If the source were to move perpendicular to the observer, one would see the frequency shift by a factor γ, i.e. the red shift would be smaller but would not vanish.

7.4 Wave Guides and Cavities

Here we consider oscillating solutions at fixed frequency. We consider the simplest case, that of a vacuum surrounded by a perfect conductor. For non-perfect conductors the boundary conditions are more complicated due to the penetration of the fields into the media. Such penetration also leads to damping of waves. Because this course explicitly ignores electromagnetism in media, our coverage of wave guides will be brief.

The boundary conditions for an electric field at a conductor are simple, $\vec{E}\times\hat{n} = 0$ at the surface. Any non-zero transverse field would cause an infinite current along the surface. The conductor has no problem with the perpendicular component being non-zero. From Gauss's law, one can see that the surface charge density, $\sigma = \vec{E}\cdot\hat{n}/4\pi$, can maintain the field being zero inside the conductor. For waves the fields are changing in time, and Maxwell's equation $\nabla\times\vec{B} = \partial_t\vec{E}$ demands that the presence of an oscillating electric field be accompanied by an oscillating magnetic field. Also, any oscillating magnetic field would be accompanied by an electric field due to the Maxwell's equation, $\nabla\times\vec{E} = -\partial_t\vec{B}$. This leads to the demand the $\vec{B} = 0$ inside the conductor – except for static fields, which are not relevant for waves. Whereas a charge density can screen the perpendicular component of $\vec{E}$, there is no such thing as magnetic charge density, $\nabla\cdot\vec{B} = 0$, so the boundary condition for magnetic field is $\vec{B}\cdot\hat{n} = 0$. One can then consider the transverse magnetic field at the surface. One can draw a loop drawn just outside the surface for a distance $\vec{L}$, then cuts into the surface, return in the opposite direction $-\vec{L}$ inside the surface, then close the loop. If there is magnetic field along $\vec{L}$ outside the loop, one can still have no magnetic field inside the conductor if there is a surface current $\vec{j}_s$. Thus, the boundary conditions for a perfect conductor are:

$$\begin{aligned}
\hat{n}\times\vec{E} &= 0, \\
\hat{n}\cdot\vec{B} &= 0, \\
\hat{n}\cdot\vec{E} &= 4\pi\sigma, \\
\hat{n}\times\vec{B} &= 4\pi\vec{j}_s.
\end{aligned} \tag{7.13}$$

First we consider a rectangular wave guide, constrained in the x and y directions

$$\begin{aligned}
0 < x < L_x, \\
0 < y < L_y.
\end{aligned} \tag{7.14}$$

The equations of motion for a wave, rewriting Eq.s (7.2) and (7.3),

$$\begin{aligned} \nabla^2\vec{E} &= \partial_t^2\vec{E}, \\ \nabla^2\vec{B} &= \partial_t^2\vec{B}, \end{aligned} \tag{7.15}$$

and viewing the boundary conditions suggest solutions of the form for the electric field,

$$\begin{aligned} E_x(\vec{r},t) &= E_{0x}e^{-i\omega t+ik_z z}\cos(q_x x)\sin(q_y y), \\ E_y(\vec{r},t) &= E_{0y}e^{-i\omega t+ik_z z}\cos(q_y y)\sin(q_x x), \\ E_z(\vec{r},t) &= E_{0z}e^{-i\omega t+ik_z z}\sin(q_y y)\sin(q_x x). \end{aligned} \tag{7.16}$$

with

$$\begin{aligned} \omega^2 &= q_x^2+q_y^2+k_z^2, \\ q_x &= \frac{n_x\pi}{L_x}, \quad q_y = \frac{n_y\pi}{L_y}, \end{aligned}$$

n_x and n_y being any four integers. In principal, one could have imagined using different numbers n_i for E_x, E_y and E_z if they somehow added up to the same frequency due to a fortuitous choice of L_x and L_y – we return to that further below. One can also write down solutions for the magnetic field wave equations,

$$\begin{aligned} B_x(\vec{r},t) &= B_{0x}e^{-i\omega t+ik_z z}\sin(q_x x)\cos(q_y y), \\ B_y(\vec{r},t) &= B_{0y}e^{-i\omega t+ik_z z}\sin(q_y y)\cos(q_x x), \\ B_z(\vec{r},t) &= B_{0z}e^{-i\omega t+ik_z z}\cos(q_y y)\cos(q_x x). \end{aligned} \tag{7.17}$$

By inspection one can see that these six components listed above have the same frequency and satisfy the boundary conditions independently, i.e. any choice of the six amplitudes E_{0i} and B_{0i} would work. However, one must also satisfy the Maxwell's equations that link $\vec{E}$ and $\vec{B}$,

$$\begin{aligned} \partial_t\vec{B} &= -\nabla\times\vec{E}, \\ \partial_t\vec{E} &= \nabla\times\vec{B}. \end{aligned} \tag{7.18}$$

This leads to the conditions

$$\begin{aligned} -i\omega B_{0x} &= ik_z E_{0y} + q_y E_{0z}, \\ -i\omega B_{0y} &= -ik_z E_{0x} - q_x E_{0z}, \\ -i\omega B_{0z} &= q_x E_{0y} - q_y E_{0x}, \\ i\omega E_{0x} &= ik_z B_{0y} + q_y B_{0z}, \\ i\omega E_{0y} &= -ik_z B_{0x} - q_x B_{0z}, \\ i\omega E_{0z} &= q_x B_{0y} - q_y B_{0x}. \end{aligned} \tag{7.19}$$

It might seem worrisome that we have six unknown amplitudes and six conditions, as we certainly expect more than one solution, not to mention an arbitary multiplicative constant. However, these conditions are not independent.

In fact, there are two independent sets of solutions, each with an arbitrary overall amplitude. The first set of solutions is referred to as transverse magnetic (TM) and the second as transverse electric (TE). For TM solutions, we set $B_{0z} = 0$ and for TE solutions we set $E_{0z} = 0$. For the TM case, we solve for the remaining amplitudes in terms of E_{0z}. The algebra gives

$$\begin{aligned}
E_{0x} &= -i\frac{q_x k_z}{\omega^2 - k_z^2} E_{0z}, \\
E_{0y} &= -i\frac{q_y k_z}{\omega^2 - k_z^2} E_{0z}, \\
B_{0x} &= -i\frac{k_z}{\omega}\frac{q_y k_z}{\omega^2 - k_z^2} E_{0z} - i\frac{q_y}{\omega} E_{0z}, \\
B_{0y} &= i\frac{k_z}{\omega}\frac{q_x k_z}{\omega^2 - k_z^2} E_{0z} + i\frac{q_x}{\omega} E_{0z}.
\end{aligned} \tag{7.20}$$

Similarly, one can solve for TE amplitudes in terms of B_{0z},

$$\begin{aligned}
E_{0x} &= -i\frac{k_z}{\omega}\frac{q_y k_z}{\omega^2 - k_z^2} B_{0z} - i\frac{q_y}{\omega} B_{0z}, \\
E_{0y} &= i\frac{k_z}{\omega}\frac{q_x k_z}{\omega^2 - k_z^2} B_{0z} + i\frac{q_x}{\omega} B_{0z}, \\
B_{0x} &= -i\frac{q_x k_z}{\omega^2 - k_z^2} B_{0z}, \\
B_{0y} &= -i\frac{q_y k_z}{\omega^2 - k_z^2} B_{0z}.
\end{aligned} \tag{7.21}$$

The group velocity is not the speed of light. From the dispersion relation (ω vs k_z) one can find the group velocity,

$$\begin{aligned}
\omega^2 &= k_z^2 + q_x^2 + q_y^2, \\
v_g &= \frac{d\omega}{dk_z} = \frac{k_z}{\omega} = \frac{k_z}{\sqrt{k_z^2 + q_x^2 + q_y^2}}.
\end{aligned} \tag{7.22}$$

The group velocity is less than the speed of light, and takes the form of a massive particle with $m = \sqrt{q_x^2 + q_y^2}$.

The procedure can be followed for any cross-sectional shape, assuming the wave-guide is translationally invariant along the z axis. One can always divide the solutions into TM and TE modes. For the TM modes, one can solve the boundary conditions first for the function $\psi(x, y)$, which gives $E_z(x, y)$ by the relation,

$$E_z = \psi(x, y)e^{-i\omega t + ik_z z}. \tag{7.23}$$

One can solve the equations for ψ from the differential equation,

$$-(\partial_x^2 + \partial_y^2)\psi = -(\omega^2 - k_z^2)\psi, \tag{7.24}$$

with the boundary conditions

$$\psi(x,y)|_S = 0. \tag{7.25}$$

This boundary condition forces E_z to be zero at the surface. Once one has solved the boundary condition, the transverse components of the electric and magnetic fields can be found via,

$$\begin{aligned} \vec{E}_t(x,y) &= \frac{ik_z}{(\omega^2-k_z^2)}e^{-i\omega t+ik_z z}\nabla_t\psi(x,y), \\ \vec{B}_t(x,y) &= \left(\frac{\omega}{k_z}\right)\hat{z}\times\vec{E}_t. \end{aligned} \tag{7.26}$$

The expression for $\vec{E}_t$ manifestly points into the surface because ψ is constant along the surface. Then, $\vec{B}_t$, which is normal to both $\vec{E}_t$ and $\hat{z}$ must lie parallel to the surface. Our next task is to show that these expressions solve each of the four Maxwell's equations. For example,

$$\nabla\times\vec{B} =? \ \partial_t\vec{E}, \tag{7.27}$$

$$\left(\frac{\omega}{k_z}\right)\nabla\times(\hat{z}\times\vec{E}_t) =? \ -i\omega\left(\vec{E}_t+\psi e^{-i\omega t+ik_z z}\hat{z}\right), \tag{7.28}$$

$$\frac{1}{k_z}\hat{z}(\nabla\cdot\vec{E}_t)-\frac{1}{k_z}\partial_z\vec{E}_t =? \ -i\left(\vec{E}_t+\psi e^{-i\omega t+ik_z z}\hat{z}\right), \tag{7.29}$$

$$\frac{\nabla_t^2\psi}{(\omega^2-k_z^2)} =? \ -\psi. \ \checkmark \tag{7.30}$$

One can check the other Maxwell equations (See HW problems).

For TE modes, a similar procedure can be applied. In that case

$$B_z = \psi(x,y)e^{-i\omega t+ik_z z}, \tag{7.31}$$

with $\psi(x,y)$ still satisfying Eq. (7.24), but with the boundary conditions

$$(\hat{n}\cdot\nabla_t)\psi(x,y)|_S = 0. \tag{7.32}$$

The transverse fields are then given by,

$$\begin{aligned} \vec{B}_t(x,y) &= \frac{ik_z}{(\omega^2-k_z^2)}e^{-i\omega t+ik_z z}\nabla_t\psi(x,y), \\ \vec{E}_t(x,y) &= -\left(\frac{\omega}{k_z}\right)\hat{z}\times\vec{B}_t. \end{aligned} \tag{7.33}$$

7.5 Homework Problems

1. Consider the Lagrangian density

$$\mathcal{L} = \frac{1}{4}F^{\mu\nu}F_{\nu\mu}-\frac{\lambda}{2}(\partial\cdot A)^2.$$

Here, the action is $S=(1/4\pi)\int d^4x\,\mathcal{L}$. The extra term here (proportional to λ) is known as the gauge-fixing term, which is a misnomer. This term explicitly destroys gauge invariance, so after one calculates quantities, one must actually use the Lorentz gauge, $\nabla\cdot A=0$ to recover the correct result.

(a) Consider the Feynmann gauge (not actually a gauge, but just a choice for λ), where $\lambda = 1$. Show that the Lagrangian, after integrating the action by parts, becomes

$$\mathcal{L} = -\frac{1}{2}\partial^\alpha A^\beta \partial_\alpha A_\beta.$$

(b) Solve for the equations of motion in the Feynmann gauge. Then, setting $\partial \cdot A = 0$, and using Eq.s (2.40), show that the equations of motion become the Maxwell's equations for free space,

$$\begin{aligned} \nabla \cdot \vec{E} &= 0, \\ \nabla \times \vec{B} &= -\partial_t \vec{E}. \end{aligned} \tag{7.34}$$

(c) (Extra Brownie Points) Write down the stress-energy tensor in the Feynmann gauge. Then, set $\partial \cdot A = 0$, show that $T_{00} = (|\vec{E}|^2 + |\vec{B}|^2)/2$. You may also need to apply equations of motion and use some messy vector identities.

2. Consider solutions for electro magnetic waves of frequency ω moving in the $\pm z$ directions which are linearly polarized in the x direction. Assume the incoming wave has a form $E_0 \hat{x} e^{-i\omega t + ikz}/2$.

   (a) Find the linear combination of such waves where the electric field vanishes at $z = 0$, i.e. reflecting off a conducting plane. Express your answer for both $\vec{E}$ and $\vec{B}$ as a real field in terms of sines and cosines.

   (b) Find the elements of the stress-energy tensor as a function of z and t.

   (c) Show that the stress-energy tensor is traceless, $T^i{}_i = 0$.

3. Consider a plane wave moving in the z direction according to Eq. (7.4) with $a_y = i a_x$ and a_x real. Taking the real part of the solution, solve for the direction of $\vec{a}$ as a function of time and position.

4. Consider a simple model of the universe where the expansion velocity for cosmological purposes is $\vec{v} = \vec{r}/t$. This corresponds to a "flat" universe with gravitational effects ignored. All matter starts at a point (the origin) and there is no acceleration for any fluid element. Observer A is moving with the source, and she records light being emitted at a time $\tau_0 = 10^5$ years after the birth of the universe, according to a clock in her pocket. A second observer, B, records the light moving past at a time $\tau = 1.4 \times 10^{14}$ years after the beginning of the universe according to a watch in his pocket. Both A and B are at rest relative to the neighboring expanding matter. If observer A records the frequency of the emitted light as being f_0, find the frequency f of the recorded light according to observer B.
   Some Help: the time measured by the co-moving observer, τ, is related to the time measured by a different observer with velocity v by the relations:

$$\tau = \frac{t}{\gamma} = t\sqrt{1 - v^2} = t\sqrt{1 - r^2/t^2} = \sqrt{t^2 - r^2}.$$

5. Show that there is no solution to the conditions for the rectangular wave-guide amplitudes in Eq. (7.19) when both E_{0z} and B_{0z} are set to zero. This demonstrates that there are no solutions other than the TE and TM solutions.

6. Show that Eq.s (7.23-7.26) satisfy the Maxwell relation $\nabla \times \vec{E} = -\partial_t \vec{B}$.

7. Consider a cylindrical wave-guide of radius R. Consider the lowest frequency TM solution to the generating function ψ satisfying the differential equation

$$\nabla_t^2 \psi(\rho, \phi) = -\alpha^2 \psi(\rho, \phi), \quad \alpha^2 = \omega^2 - k_z^2,$$

where k_z is the wavenumber for longitudinal motion.

   (a) Find a solution for ψ in polar coordinates. Express answer in terms of a_0, the first zero of the Bessel function J_0.

   (b) Find expressions for the electric and magnetic fields.

   (c) What is the group velocity of a wave with momentum k_z.

8. Consider two infinite parallel plates with the plane of the plates being along the x direction and the separation being L_x, i.e. a rectangular wave guide with $L_y = \infty$. Consider a wave moving in the z direction with wave number k_z. Using the method of generating functions,

   (a) Solve for the lowest frequency TM wave. Find expressions for the fields and the group velocity.

   (b) Solve for the lowest frequency TE wave. Again find expressions for the fields and the group velocity.

8 Radiation

Here we discuss classical radiation. The source of such radiation is the $\boldsymbol{J} \cdot \boldsymbol{A}$ terms in the Lagrangian, and the "classical" assumption is that the current does not change due to the radiated photon. In contrast, quantum emission involves changing discrete levels. For instance one could fall from a $\boldsymbol{p}$-state to an $\boldsymbol{s}$-state, with the frequency determined by the change in energy levels. For classical emission the frequency is a property of how charge moves within the source, and it is assumed that there is no feedback from the radiation that alters the current. Additionally, the classical assumption ignores the fact that light of a given frequency carries quantized amount of energy, but this latter part of the assumption is irrelevant if one records many photons.

8.1 Coupling the Electromagnetic Field to Dynamic Currents and Charges

To incorporate dynamics into the generation of electric and magnetic fields, we consider Maxwell's equations for electric and magnetic fields, written in term of the four-vector potential $\boldsymbol{A}^\mu$,

$$\begin{aligned} \partial_\alpha F^{\alpha\beta} &= 4\pi J^\beta, \\ \partial_\alpha(\partial^\alpha A^\beta - \partial^\beta A^\alpha) &= 4\pi J^\beta, \\ \partial^2 A^\beta - \partial^\beta(\partial \cdot A) &= 4\pi J^\beta. \end{aligned} \tag{8.1}$$

Similar as to our derivations for magneto-statics, this last expressions could be treated as four separate Poisson's equations if the second term of the bottom expression would disappear. Again, we argue that this term can be ignored because one can choose a gauge that explicitly cancels it, and given that physical results cannot depend on the gauge, that term cannot matter. The choice of Λ,

$$A^\alpha_{(\mathrm{new})} = A^\alpha_{(\mathrm{old})} + \partial^\alpha \Lambda, \tag{8.2}$$

must be able to enforce $\partial \cdot A_{(\mathrm{new})} = 0$.

$$\begin{aligned} \partial \cdot A_{(\mathrm{new})} &= \partial \cdot A_{(\mathrm{old})} + \partial^2 \Lambda = 0, \\ \partial^2 \Lambda &= -\partial \cdot A_{(\mathrm{old})}. \end{aligned} \tag{8.3}$$

Just as before, for any scalar function $\nabla \cdot A_{(\mathrm{old})}$, one can find a solution to Poisson's equation for Λ where $\partial \cdot A_{(\mathrm{old})}$ serves as a source. Physics doesn't depend on Λ, so we choose Λ to satisfy the gauge constraint, and are thus left with simpler equations. This is know as the Lorentz gauge, and aside from being convenient, has an attractiveness due to its invariance to boosts or rotations,

$$\partial \cdot A = 0. \tag{8.4}$$

We are thus left with four independent equations,

$$\partial^2 A^\alpha = 4\pi J^\alpha. \tag{8.5}$$

When the right-hand side was set to zero this is known as Laplace's equation, when the right-hand side is non-zero, and independent of $\boldsymbol{A}$, it is Poisson's equation. The right-hand side can be thought of as a source term. If there are no fields at $t \rightarrow -\infty$, any subsequent appearance of fields is due to the source term. The equation is linear in A, so one can write the solutions as a sum (integral) over solutions from each differential contribution on the right-hand side. If the right-hand side were,

$$\partial^2 G(x) = \delta^4(x), \tag{8.6}$$

and if the solution $G_>$ satisfied the boundary conditions of being zero for all negative times, one can apply this solution to write solutions for any form on the right-hand side. For an arbitrary source function, $S(x)$, where one is searching for a solution $F(x)$,

$$\partial^2 F(x) = S(x), \tag{8.7}$$

one could write the solution for F as

$$F(x) = \int_{x_0'<x_0} d^4x'\; S(x')G_R(x-x'), \tag{8.8}$$

which effectively is taking a linear combination of solutions for each differential source element to generate a solution for a continuous source. For the case here, the source function for solving for the evolution of $A^\alpha(r)$ is $4\pi J^\alpha(r')$, and

$$A^\alpha(x) = 4\pi \int_{x_0'<x_0} d^4x'\; J^\alpha(x')G_R(x-x'). \tag{8.9}$$

The function $G_R(r)$, the retarded Green's Function, is independent of the source, and applies to any Poisson equation. The retarded solution is the solution to the differential equation in Eq. (8.6) that vanishes for negative times.

The Green's function is found by solving Eq. (8.6). By inspection, one can see that

$$G(t,\vec{r}) = \frac{-1}{(2\pi)^4}\int d\omega d^3k\, \frac{e^{i\omega t - i\vec{k}\cdot\vec{r}}}{\omega^2-k^2} \tag{8.10}$$

is a solution to Eq. (8.6). However, this solution does not satisfy the boundary conditions that it vanish for all negative times. Altering the solution to satisfy the boundary condition,

$$G_R(t,\vec{r}) = \frac{-1}{(2\pi)^4}\int d\omega d^3k\, \frac{e^{i\omega t - i\vec{k}\cdot\vec{r}}}{(\omega-k-i\epsilon)(\omega+k-i\epsilon)}, \quad \epsilon\rightarrow 0+. \tag{8.11}$$

Performing the integral over ω by contour integration,

$$\begin{aligned} G_R(t,\vec{r}) &= \frac{-i}{(2\pi)^3}\int d^3k\, \frac{1}{2k}\left(e^{ikt-i\vec{k}\cdot\vec{r}} - e^{-ikt-i\vec{k}\cdot\vec{r}}\right)\Theta(t) \\ &= \frac{1}{(2\pi)^3}\int d^3k\, \frac{1}{k}\sin(kt)e^{-i\vec{k}\cdot\vec{r}}\Theta(t). \end{aligned} \tag{8.12}$$

Next, choosing θ as the angle between $\vec{r}$ and $\vec{k}$, writing $e^{i\vec{k}\cdot\vec{r}} = e^{ikr\cos\theta}$, and integrating over $\cos\theta$ then k,

$$\begin{aligned} G_R(t,\vec{r}) &= \frac{1}{4\pi^2 r}\int_{-\infty}^{\infty} dk\;\sin(kt)\sin(kr)\Theta(t) \\ &= \frac{1}{4\pi r}\delta(t-r). \end{aligned} \tag{8.13}$$

This allows the vector potential to be written in terms of an integral driven by currents at previous times,

$$A^\alpha(x) = \int d^4x' \, \frac{1}{|\vec{x}-\vec{x}'|} J^\alpha(x')\delta(x_0 - x_0' - |\vec{x}-\vec{x}'|). \tag{8.14}$$

The physical interpretation of this is clear, the vector potential is driven by the configuration of the currents along the light-front. For the static case, J^α does not depend on time, one quickly recovers the usual expressions given in the previous sections (See HW problem).

8.2 Radiation from an Accelerating Point Charge

Radiative energy flux is given by the Poynting vector, $\vec{S} = \vec{E} \times \vec{B}/4\pi$, and because for static cases $\vec{E}$ and $\vec{B}$ fall off at least as quickly as $1/r^2$, the flux must fall off faster than $1/r^2$ and thus there is no radiated energy. However, this changes for the case of accelerated charges because the current density depends on time. . We consider the equation for vector potential, Eq. (8.14), and consider a moving point charge of charge e. In that case,

$$d^4x' J^\alpha(x') = eu^\alpha d\tau, \tag{8.15}$$

when the differential covers the particle's position. Here, $d\tau$ is the differential time as measured in the frame of the charge. To verify this, consider the frame of the particle, $u = (1,0,0,0)$. In that frame $J^0 d^3x = e, \vec{J}d^3x = 0$, which is the correct answer, and because the Lorentz indices match, d^4x is a scalar, it must be correct in all frames.

The vector potential is then

$$A^\alpha(x) = 2e \int d\tau \; u^\alpha(\tau)\delta[(x-r(\tau))^2]\Theta(x_0 - r_0), \tag{8.16}$$

where $r^\alpha(\tau)$ is the trajectory of the particle. The extra factor of 2 and stepfunction, and the missing $1/|x \vec{-} r|$ arise from the fact that the delta function is now a function of the invariant distance squared. Given that $(d/dx)\delta(f(x)) = \delta(f(x))/f'$, the expressions are equivalent. Because the new delta function would also contribute when $r_0 > x_0$, the step function is added. To find the electro-magnetic fields,

$$\begin{aligned}
\partial^\alpha A^\beta &= 2e \int d\tau \; u^\beta \partial^\alpha \delta[(x-r(\tau))^2] \qquad (8.17)\\
&= 2e \int d\tau \; u^\beta \frac{\partial \delta[(x-r(\tau))^2]}{\partial \tau} \frac{\partial \tau}{\partial (x-r(\tau))^2} \partial^\alpha (x-r(\tau))^2 \\
&= -2e \int d\tau u^\beta \frac{(x-r)^\alpha}{(r-x)\cdot u} \frac{\partial \delta[(x-r(\tau))^2]}{\partial \tau} \\
&= 2e \int d\tau \; \delta[(x-r(\tau))^2] \partial_\tau \left\{ \frac{u^\beta(\tau)(r(\tau)-x)^\alpha}{u(\tau)\cdot(r(\tau)-x)} \right\}.
\end{aligned}$$

Here, we have made use of the fact that $u^\alpha = dr^\alpha/d\tau$. We are only interested in the fields far away, $x_0 \approx |\vec{x}| >> r_i$. Thus, we need only worry about derivatives of u above because derivatives of x would result in additional factors of $1/x$. Taking the derivatives, then throwing away all terms that fall off too quickly,

$$\begin{aligned} \partial^\alpha A^\beta(x) &\approx 2e \int d\tau\, \delta[(x-r(\tau))^2] \left\{ \frac{x^\alpha a^\beta}{(u\cdot x)} a^\beta - \frac{x^\alpha u^\beta}{(u\cdot x)^2}(a\cdot x) \right\}, \\ a^\beta &\equiv \frac{d}{d\tau} u^\beta. \end{aligned} \tag{8.18}$$

The zero$^{\rm th}$ component a can be rewritten (see HW problem) as $a^0 = \vec{u}\cdot\vec{a}/u_0$. One can now again use the chain rule for delta functions to obtain

$$\partial^\alpha A^\beta(x) = \frac{e}{(u\cdot x)^2} x^\alpha \left(a^\beta - \frac{u^\beta(a\cdot x)}{(u\cdot x)} \right). \tag{8.19}$$

We are interested in radiation, which implies the long-distance limit where $x \to \infty$, so we replaced the $\approx$ sign in Eq. (8.18) with an equal sign and replaced $u\cdot(x-r)$ with $u\cdot x$.

The electric and magnetic fields can now be calculated,

$$E^i = \frac{e}{(u\cdot x)^2} \left\{ x^0 \left(a^i - \frac{u^i(a\cdot x)}{(u\cdot x)} \right) - x^i \left(a^0 - \frac{u^0(a\cdot x)}{(u\cdot x)} \right) \right\}. \tag{8.20}$$

With a significant amount of algebraic effort, one can express this in the form,

$$\vec{E} = e \left\{ \frac{\hat{n}\times[(\hat{n}-\vec{\beta})\times\dot{\vec{\beta}}]}{(1-\vec{\beta}\cdot\hat{n})^3|\vec{x}|} \right\}. \tag{8.21}$$

Here, $\hat{n}$ is a unit vector in the direction of $\vec{x}$, and $\vec{\beta} = \vec{u}/u_0 = \vec{v}/c$ is the non-relativistic velocity. Again, with a bit of effort, one can see that the magnetic field is given by

$$\vec{B} = \hat{n}\times\vec{E}. \tag{8.22}$$

The direction of $\vec{E}$ defines the polarization. For small velocities that direction can be found by taking the direction of $\dot{\vec{\beta}}$, then projecting out the part of the vector along $\vec{x}$. The small velocity limit, $\beta << 1$, also makes it easy to calculate the net power. The power per area is given by the magnitude of the Poynting vector. For $\beta << 1$,

$$\begin{aligned} |\vec{S}| &= \frac{1}{4\pi}|\vec{E}|^2 \\ &= \frac{e^2}{4\pi|\vec{x}|^2}|\dot{\vec{\beta}}|^2 \sin^2\theta, \end{aligned} \tag{8.23}$$

where θ is the angle between $\hat{n}$ and $\vec{\beta}$. Integrating over the area, $dA = 2\pi|\vec{x}|^2 d\cos\theta$, the net power is

$$P = \frac{2e^2}{3c}|\dot{\vec{\beta}}|^2, \tag{8.24}$$

which is known as Larmor's formula. The factor $1/c$ was added should you need to calculate in units where $c \neq 1$.

For relativistic motion, the expression is more complicated. In Eq. (8.21) one cannot ignore the $\vec{\beta}$ in the numerator, but more dramatically, the denominator,

$$\frac{1}{(1-\vec{\beta}\cdot\hat{n})^3}, \tag{8.25}$$

diverges as $\beta \to 1$ when the emission direction $\vec{x}$ is parallel to the velocity $\vec{\beta}$. This is especially true for circular motion of electrons in a high energy accelerator. Due to their light mass β is very nearly unity. The acceleration is inward, toward the center of the motion, so one can use Eq. (8.21) to see that the emitted light is linearly polarized with $\vec{E}$ perpendicular to both the acceleration and the direction of the light, which tends to be parallel to the velocity. Thus, that light is polarized perpendicular to the plane of the motion.

8.3 Liénard-Wiechert Potentials

The expressions for the fields due to currents in the previous section assumed that the observer was far away, in both distance and time, from the fields. A more general expression for the vector potential was given in Eq. (8.16),

$$\begin{aligned} A^\alpha(x) &= 2e\int d\tau\, u^\alpha(\tau)\delta[(x-r(\tau))^2]\Theta(x_0-r_0), \\ &= \frac{eu^\alpha(\tau)}{u\cdot(x-r(\tau))}. \end{aligned} \tag{8.26}$$

Here $r(\tau)$ is the position of the particle at a time when a light-pulse from the particle's trajectory would reach space-time point x. Effectively, one would follow the trajectory until one found such a point. Because the particle moves slower than the speed of light, there can be only one such point. The factor $1/(u\cdot(x-r))$ can be factored into

$$\begin{aligned} u\cdot(x-r) &= u_0(x_0-r_0)\,(1-\hat{n}\cdot\vec{v})\,, \\ A^\alpha(x) &= \frac{e(u^\alpha(\tau)/u_0)}{1-\hat{n}\cdot\vec{v}}, \end{aligned} \tag{8.27}$$

where $\hat{n}$ is the unit vector parallel to $\vec{x}-\vec{r}$. As was seen in the last section, in the non-relativistic limit this factor simply provides the inverse distance from the point of emission to the observer. However, relativistically the additional factor $(1-\hat{n}\cdot\vec{v})^{-1}$ amplifies the response of the potential to the charge when the velocity approaches the speed of light and is pointed toward the observer.

In a simulation of relativistic charged particles interacting electromagnetically, Liénard-Wiechert potentials offer one way (probably not the most computationally efficient way) of including retardation effects. Rather than assuming instantaneous interactions, where the potentials depend on the current positions and currents of the particles, one would calculate the contribution to the $A^\alpha(x)$ for some point x by using the positions and velocities of each charge at the appropriate retarded time.

8.4 Radiation from relativistic particles

The velocity of a proton in the LHC is 99.9999991% the speed of light. Electrons are lighter, so even though the energy of the LEP ring of the LHC is smaller than that of the LHC, the speed of an electron in LEP is 99.9999999988% the speed of light. In such cases the relativistic factors are surprisingly large and non-trivial.

Using the results of Eq.s (8.21) and (8.22), the power per solid angle from an accelerating charge is

$$\frac{dP}{d\Omega} = \frac{e^2}{4\pi(1-\vec{\beta}\cdot\hat{n})^6}|(\hat{n}-\vec{\beta})\times\dot{\vec{\beta}})|^2. \tag{8.28}$$

This represents the energy emitted into the solid angle during a time dx_0. During that time interval dx_0, the time interval of the particle's trajectory that contributed to the emission is

$$\begin{aligned}
dr_0 &= dx_0\frac{dr_0}{dx_0}, \qquad (8.29)\\
(r-x)^2 &= 0,\\
2(r_0-x_0)(dr_0-dx_0)-2(\vec{r}-\vec{x})\cdot d\vec{r} &= 0,\\
dr_0\left[r_0-x_0-\vec{\beta}\cdot(\vec{r}-\vec{x})\right] &= dx_0(r_0-x_0),\\
\frac{dr_0}{dx_0} &= \frac{1}{(1-\vec{\beta}\cdot\hat{n})}.
\end{aligned}$$

The energy emitted per time interval of the accelerated charge is thus

$$\frac{dP}{d\Omega} = \frac{e^2}{4\pi(1-\vec{\beta}\cdot\hat{n})^5}|\hat{n}\times[(\hat{n}-\vec{\beta})\times\dot{\vec{\beta}})]|^2. \tag{8.30}$$

Here, θ is the angle between $\vec{\beta}$ and $\hat{n}$. With a devoted effort, one can integrate the differential expression for power. The result for the remarkably difficult integral is

$$P = \frac{2}{3c}e^2\gamma^6\left[\dot{\beta}^2-|\vec{\beta}\times\dot{\vec{\beta}}|^2\right]. \tag{8.31}$$

According to Jackson, this is the Liénard result from 1898, which is remarkable given that special relativity was not generally explained until 1905.

First, we consider the case where $\vec{\beta}$ and $\dot{\vec{\beta}}$ are parallel or anti-parallel. This would be the case for a linear accelerator or for a charge gradually stopping. In that case

$$\frac{dP}{d\Omega} = \frac{e^2}{4\pi(1-\beta\cos\theta)^5}|\dot{\vec{\beta}}|^2\sin^2\theta. \tag{8.32}$$

The unusual angular shape is solely determined by the magnitude of the velocity β. It vanishes in the forward direction due to the $\sin^2\theta$ factor, then has a maxima at an angle $\theta_{\rm max} < \pi/2$. In the $\beta << 1$ limit, the maximum $\theta_{\rm max} \approx \pi/2$, and in the ultra-relativistic limit the maximum $\theta_{\rm max} \approx 0$ and the emission is strongly forward-peaked. The angular shape is identical for

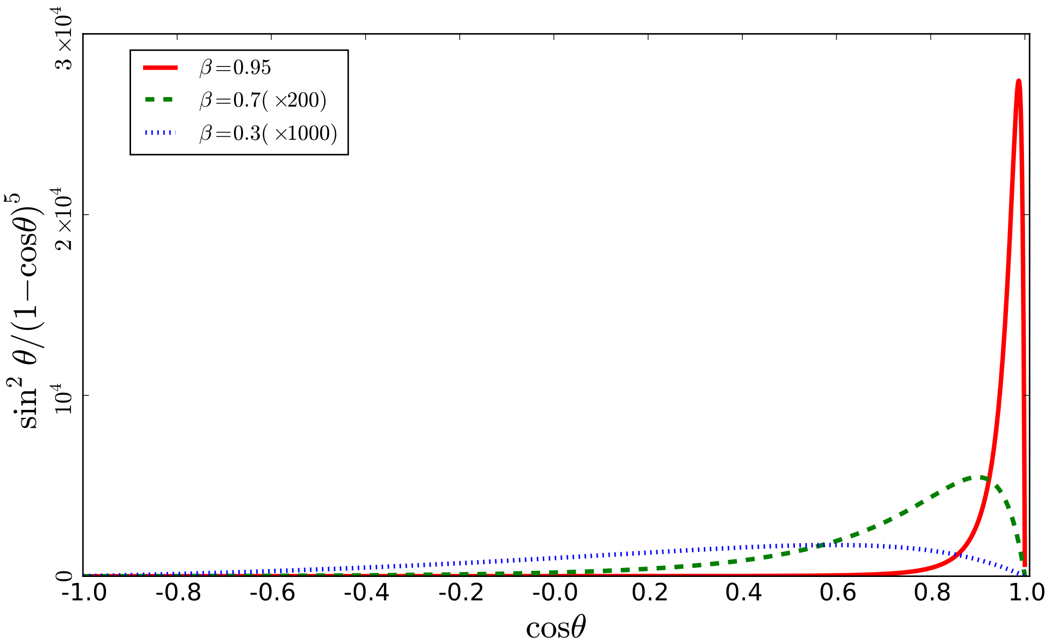


Figure 8.1: The angular dependence from Eq. (8.32) is shown for three velocities. As β approaches the speed of light, the strength of the radiation increases dramatically and the emission becomes strongly forward-peaked. For the same acceleration, radiation increases by more than three orders of magnitude as β increases from 0.5 to 0.95.

accelerating or decelerating particles, and is peaked in the direction of $\vec{\beta}$, not in the direction of $\dot{\vec{\beta}}$. From Eq. (8.31) one can see,

$$P = \frac{2e^2\dot{\beta}^2}{3c}\gamma^6. \tag{8.33}$$

Another interesting example is for when $\vec{\beta}$ is perpendicular to the acceleration $\dot{\vec{\beta}}$, as is the case for circular motion. In this case Eq. (8.30) becomes

$$\frac{dP}{d\Omega} = \frac{e^2}{4\pi(1-\beta n_\beta)^5}|\dot{\vec{\beta}}|^2\left((1-\beta n_\beta)^2-(1-\beta^2)n_r^2\right). \tag{8.34}$$

Here, n_β is the component of n parallel to β and n_r is the component parallel to $\dot{\vec{\beta}}$. As $n_\beta \to 1$, n_r vanishes and the angular dependence scales as $1/(1-\beta)^3$. Unlike the form for the case where the acceleration and velocity were parallel, this does not vanish as $\theta \to 0$, and becomes infinite in the limit that $\beta \to 1$. For this reason, electron accelerators are excellent candidates for high-luminosity light sources. From Eq. (8.31) the net power is then

$$P = \frac{2}{3c}e^2\dot{\beta}^2\gamma^4. \tag{8.35}$$

This differs from the case where $\vec{\beta}$ is parallel to $\dot{\vec{\beta}}$ by two powers of γ.

8.5 Frequency Dependence of Radiation from Accelerated Charge

For some trajectory, $\vec{r}(\tau)$, one can calculated the contribution to the electric field from Eq. (8.20), $\vec{E}(x)$, and then find the Fourier transform of the electric field, $\tilde{\vec{E}}(\vec{x},\omega)$. From this, one can calculate the energy radiated per unit frequency at some large distance R and angle Ω,

$$\begin{aligned}
\frac{dU}{d\Omega} &= \frac{1}{4\pi}R^2\int dt\,\vec{E}(R,t)^2 \qquad (8.36)\\
&= \frac{1}{4\pi(2\pi)^2}R^2\int dtd\omega d\omega'\,\tilde{\vec{E}}(\vec{R},\omega)\tilde{\vec{E}}^*(\vec{R},\omega')e^{i(\omega-\omega')t}\\
&= \frac{1}{8\pi^2}R^2\int d\omega\,\left|\tilde{\vec{E}}(\vec{R},\omega)\right|^2,\\
&= \frac{1}{4\pi^2}R^2\int_0^\infty d\omega\,\left|\tilde{\vec{E}}(\vec{R},\omega)\right|^2\\
\tilde{\vec{E}}(\vec{R},\omega) &\equiv \int dt\,e^{i\omega t}\vec{E}(\vec{R},\omega).
\end{aligned}$$

Now, using Eq. (8.20) to express $\tilde{\vec{E}}$,

$$\begin{aligned}
\tilde{\vec{E}}(\omega) &= \frac{e}{R}\int dt\,e^{i\omega t}\left[\frac{\hat{n}\times[\hat{n}-\vec{\beta}]\times\dot{\vec{\beta}}}{(1-\vec{\beta}\cdot\hat{n})^3}\right]_{\rm ret}, \qquad (8.37)\\
&= \frac{e}{R}\int dt' e^{i\omega(t'+R-\hat{n}\cdot\vec{r}(t'))}\left[\frac{\hat{n}\times[\hat{n}-\vec{\beta}]\times\dot{\vec{\beta}}}{(1-\vec{\beta}\cdot\hat{n})^2}\right].
\end{aligned}$$

The third step shortened the integral over all frequencies to simply those over positive frequencies by noting that $|\tilde{\vec{E}}(\omega)|^2$ is the same for ω and $-\omega$. The first expression involves calculating the velocities at the retarded times but the integral is over the observers time, whereas the second expression involves replacing the integral over the observer's time with an integral over the retarded time, t',

$$\begin{aligned}
t &= t'+R-\hat{n}\cdot\vec{r}(t'),\quad R\equiv|\vec{x}|, \qquad (8.38)\\
dt/dt' &= 1-\vec{\beta}\cdot\hat{n}.
\end{aligned}$$

which should be accurate for large R. For the next step we use an identity (See H.W. problem),

$$\frac{\hat{n}\times(\hat{n}-\hat{\beta})\times\dot{\vec{\beta}}}{(1-\vec{\beta}\cdot\vec{n})^2} = \frac{d}{dt}\left[\frac{\hat{n}\times(\hat{n}\times\vec{\beta})}{1-\vec{\beta}\cdot\hat{n}}\right]. \qquad (8.39)$$

Inserting this into Eq. (8.37) allows us to express $\tilde{\vec{E}}(\omega)$ in terms of an integral involving only the velocity,

$$\begin{aligned}
\tilde{\vec{E}}(\omega) &= \frac{e}{R}\,e^{i\omega(t'+R-\hat{n}\cdot\vec{r}(t'))}\frac{\hat{n}\times(\hat{n}\times\vec{\beta})}{1-\vec{\beta}\cdot\hat{n}}\Bigg|_{t'=t_i}^{t'=t_f} \qquad (8.40)\\
&\quad -i\frac{e}{R}\omega\int_{t_i}^{t^f} dt'\,e^{i\omega(t'+R-\hat{n}\cdot\vec{r}(t'))}\left[\hat{n}\times(\hat{n}\times\vec{\beta})\right].
\end{aligned}$$

One can add a factor $e^{-\epsilon|t|}$ to each integrand, which makes it possible to discard the first term. Effectively this term represents slowing down the currents at large times with infinitesimally slow accelerations, which doesn't cause any radiation, but allows one to regulate the integrals. Thus, for the radiative energy,

$$\frac{dU}{d\omega d\Omega} = \frac{e^2\omega^2}{4\pi^2}\left|\int dt'\, e^{i\omega(t'-\hat{n}\cdot\vec{r}(t'))}\left[\hat{n}\times(\hat{n}\times\vec{\beta})\right]\right|^2. \tag{8.41}$$

8.6 Radiation from Oscillating Systems with Well-Defined Frequencies

All currents can be expressed in terms of Fourier transforms,

$$J^\alpha(x) = \frac{1}{2\pi}\int d\omega\, \tilde{J}^\alpha(\omega,\vec{x})e^{-i\omega x_0}. \tag{8.42}$$

Each frequency component contributes to the vector potential. Equation 8.14 becomes

$$\begin{aligned} A^\alpha(\vec{x},t) &= \frac{1}{2\pi}\int d\omega d^3x' dt'\, e^{-i\omega t'}\frac{\tilde{J}^\alpha(\omega,\vec{x}')}{|\vec{x}-\vec{x}'|}\delta(t'+|\vec{x}-\vec{x}'|-t) \\ &= \frac{1}{2\pi}\int d\omega d^3x'\, e^{i\omega(|\vec{x}-\vec{x}'|-t)}\frac{\tilde{J}^\alpha(\omega,\vec{x}')}{|\vec{x}-\vec{x}'|}. \end{aligned} \tag{8.43}$$

Any classically radiating system could be treated using the Fourier transform. However, this does not always simplify the problem. If the $\tilde{J}$ has a range of frequencies, they all contribute to the vector potential, and the intensities, which require squaring the fields, will likely involve integrals over two frequencies, $d\omega d\omega'\tilde{J}(\omega)\tilde{J}(\omega')\cdots$.

However, the expressions simplify significantly when only one well defined frequency enters the problem. In that case we forego using Fourier transforms, and instead assume that the time dependence of $J(x)$ factors into a single phase $e^{-i\omega_0 t}$,

$$J^\alpha(x) = e^{-i\omega_0 t}j^\alpha(\vec{x}). \tag{8.44}$$

This would be reasonable for well-designed antennas, or some other system driven by an oscillating term. Some systems have a characteristic frequency, e.g. the orbital frequency in a synchotron, but the Fourier transform would include many harmonics of that motion due to the fact that the current of a single electron looks like a series of delta function pulses separated by the orbital period. Note j_0 is determined by $\vec{j}$ through current conservation.

$$i\omega_0 j_0(\vec{x}) = \nabla\cdot\vec{j}(\vec{x}). \tag{8.45}$$

In this case the solution for A from Eq. (8.14) becomes

$$\begin{aligned} A^\alpha(\vec{x},t) &= e^{-i\omega_0 t}\int d^3x'\, e^{i\omega_0(|\vec{x}-\vec{x}'|)}\frac{j^\alpha(\vec{x}')}{|\vec{x}-\vec{x}'|} \\ &= e^{-ikt}\int d^3x'\, e^{ik(|\vec{x}-\vec{x}'|)}\frac{j^\alpha(\vec{x}')}{|\vec{x}-\vec{x}'|}, \end{aligned} \tag{8.46}$$

where $k = \omega_0/c$ is the wave number for light with frequency ω_0. Here, we switch the notation from ω_0 to k to emphasize that for low frequencies we can expand about kx' being a small number.

We are interested in radiation, $r \to \infty$, so we can expand

$$|\vec{x} - \vec{x}'| \approx r - \hat{n} \cdot \vec{x}', \tag{8.47}$$

where once again $\hat{n}$ points in the direction of $\vec{r} \equiv \vec{x}$. For radiation, we only wish terms that fall of as $1/r$, so the Poynting vector falls as $1/r^2$, so

$$A^\alpha(\vec{x}, t) = \frac{e^{ik(r-t)}}{r} \int d^3x' \, j^\alpha(\vec{x}) e^{-ik\hat{n}\cdot\vec{x}'}. \tag{8.48}$$

For slower frequencies one can expand $e^{-i\hat{n}\cdot\vec{x}'}$ in powers of k. This expansion converges well if the period of the oscillating source is much longer than the time it takes light to cross the source.

$$A^\alpha(\vec{x}, t) = \frac{e^{ik(r-t)}}{r} \sum_n \frac{(-ik)^n}{n!} \int d^3x' \, j^\alpha(\vec{x}')(\hat{n} \cdot \vec{x}')^n. \tag{8.49}$$

8.7 Electric and Magnetic Dipole Radiation

If one keeps the lowest term in the expansion of Eq. (8.49),

$$\begin{aligned}
\vec{A}(\vec{x}, t) &= \frac{e^{ik(r-t)}}{r} \int d^3x' \, \vec{j}(\vec{x}'), \\
A_i(\vec{x}, t) &= \frac{e^{ik(r-t)}}{r} \int d^3x' \, j_k(\vec{x}')(\partial_k x'_i) \\
&= -\frac{e^{ik(r-t)}}{r} \int d^3x' \, x'_i(\nabla \cdot \vec{j}) \\
\vec{A}(\vec{x}, t) &= -\frac{e^{ik(r-t)}}{r} \int d^3x' \, \vec{x}(\nabla \cdot \vec{j}(\vec{x}')) \\
&= -ik\frac{e^{ik(r-t)}}{r} \int d^3x' \, \vec{x}' j_0(\vec{x}') \\
&= -ik\frac{e^{ik(r-t)}}{r} \vec{p} \\
\vec{p} &= \int d^3x' \, \vec{x}' j_0(\vec{x}').
\end{aligned} \tag{8.50}$$

After factoring out the $e^{-i\omega t}$ factor from $\rho(\vec{x}, t)$, $\vec{p}$ looks like the electric dipole moment. Here, the second step exploited current conservation. The magnetic and electric fields then become

$$\begin{aligned}
\vec{B} &= k^2(\hat{n} \times \vec{p})\frac{e^{ik(r-t)}}{r}, \\
\vec{E} &= k^2(\vec{p} - \hat{n}(\vec{n} \cdot \vec{p}))\frac{e^{ik(r-t)}}{r}.
\end{aligned} \tag{8.51}$$

The power per solid angle is then

$$\begin{aligned}\frac{dP}{d\Omega} &= \frac{1}{8\pi}\vec{E}\times\vec{B} \\ &= \frac{1}{8\pi}k^2|\hat{n}\times\vec{p}|^2.\end{aligned} \tag{8.52}$$

The factor of $1/8\pi$, instead of the usual $1/4\pi$ expression for the Poynting vector arises because one uses only the real part of the fields, and the average of $\cos^2[k(r-t)]$ is one half. Finally, one can integrate the flux to find the net power radiated,

$$\begin{aligned}P &= \frac{1}{8\pi}k^2p^2\int d\Omega\,\sin^2\theta \\ &= \frac{k^4}{3}|\vec{p}|^2.\end{aligned} \tag{8.53}$$

This is referred to as electric dipole radiation.

To extract the magnetic dipole moment one can look at the next power ($n=1$) in the expansion in Eq. (8.49). This is one higher power in k,

$$\vec{A} = \frac{e^{ik(r-t)}}{r}(-ik)\int d^3x'\,\vec{j}(\vec{x}')(\hat{n}\cdot\vec{x}'). \tag{8.54}$$

For the next step we make use of a vector identity

$$\vec{A}(\vec{B}\cdot\vec{C}) = \frac{1}{2}\vec{A}(\vec{B}\cdot\vec{C}) + \frac{1}{2}\vec{B}(\vec{A}\cdot\vec{C}) + \frac{1}{2}\vec{C}\times(\vec{A}\times\vec{B}). \tag{8.55}$$

With this identity,

$$\begin{aligned}\int d^3x'\,\vec{j}(\vec{x}')(\hat{n}\cdot\vec{x}') &= \frac{1}{2}\int d^3x'\,\hat{n}\times[\vec{x}'\times\vec{j}(\vec{x}')] \\ &+\frac{1}{2}\int d^3x'\,\left[\vec{j}(\vec{x}')(\hat{n}\cdot\vec{x}')+\vec{x}'(\hat{n}\cdot\vec{j}(\vec{x}')\right].\end{aligned} \tag{8.56}$$

As was done for the electric dipole case, the second term can be written in terms of the charge density by applying current conservation and integrating by parts.

$$\begin{aligned}\int d^3x'\,\vec{j}(\vec{x}')(\hat{n}\cdot\vec{x}') &= \frac{1}{2}\int d^3x'\,\hat{n}\times[\vec{x}'\times\vec{j}(\vec{x}')] \\ &-\frac{ik}{2}\int d^3x'\,\vec{x}'(\hat{n}\cdot\vec{x}')j_0(\vec{x}').\end{aligned} \tag{8.57}$$

The second term involves second moments of $\vec{x}'$ and leads to electric quadrupole radiation. As such, it is of a higher power of k. The first term is the magnetic dipole term,

$$\begin{aligned}\vec{A} &= (-ik)\frac{e^{ik(r-t)}}{r}\vec{m}\times\vec{n} \\ \vec{m} &= \frac{1}{2}\int d^3x'\,\vec{x}'\times\vec{j}(\vec{x}').\end{aligned} \tag{8.58}$$

Even though the magnetic dipole contribution as one higher power in k from the expansion in Eq. (8.49) compared to the electric dipole term, both terms came out linear in k. This is because the use of current conservation, $ik\rho = \nabla\cdot\vec{j}$, added an additional factor of k to the electric dipole term. Calculating the electric and magnetic fields,

$$\begin{aligned} \vec{B}(\vec{r},t) &= k^2(\vec{m} - \vec{n}(\vec{n}\cdot\vec{m}))\frac{e^{ik(r-t)}}{r}, \\ \vec{E}(\vec{r},t) &= -k^2(\vec{n}\times\vec{m})\frac{e^{ik(r-t)}}{r}. \end{aligned} \tag{8.59}$$

For the electric dipole case, the polarization is defined by the electric field pointing along the direction of $\vec{p}$ after the $\hat{n}$ component was projected away. For magnetic dipole radiation, the direction of the magnetic field in the wave is along the direction of $\vec{m}$ after the $\hat{n}$ component is projected away.

Electric dipole radiation can come from having charge oscillating back and forth along a wire. It also ensues from having a charge move in a circle. One would think that a particle moving in a circle would have magnetic dipole radiation because $\vec{x}'\times\vec{J}$ is non-zero. However, this is constant in time for circular motion, so there is no finite frequency component, and thus radiation from circular motion proceeds through the electric dipole form.

8.8 Homework Problems

1. Consider Eq. (8.14) in the case where J^α has no time dependence. Show that one quickly obtains the usual expressions for the potentials in the static case.

2. Using the fact that $\nabla^2(1/r) = -4\pi\delta^3(\vec{r})$,

   (a) show that any function $f(r-t)$ satisfies the differential equation,

   $$\partial^2\left(\frac{f(r-t)}{r}\right) = 4\pi f(r-t)\delta^3(\vec{r}).$$

   (b) Now, let $f(r-t) = \delta(r-t)$. Show that this satisfies the equation

   $$\partial^2\left(\frac{f(r-t)}{r}\right) = 0$$

   for all $t > 0$. Also, because $r > 0$ the function is zero for $t < 0$.

   (c) Show that the form $f(r-t) = \delta(r-t)$ satisfies the integral of Eq. (8.7).

   $$\int_{-\epsilon}^{\epsilon} dt\,\left[\partial^2\left(\frac{\delta(r-t)}{r}\right)\right] = -4\pi\int dt\,\delta^4(x).$$

3. Consider a function $f(x)$ that is a super-position of plane waves,

   $$f(x) = \int dk\, g(k')e^{i\omega(k')t - ik'x + i\phi_0(k')},$$

where $g(k')$ is a narrow function centered about k, e.g.

$$g(k') = \frac{1}{\sqrt{2\pi a^2}} e^{-(k'-k)^2/2a^2},$$

with $a << k$.

(a) For a given time t find the position x at which the phase $[i\omega(k')t - ik'x + i\phi_0(k')]$ is steady as a function of k' at $k' = k$, i.e.

$$\frac{d}{dk'}[i\omega(k')t - ik'x + i\phi_0] = 0.$$

(b) What are the group velocities for the following cases:
a) massless particle in a vacuum, $\omega = |k|c$
b) massive particles in a vacuum, $\hbar\omega = (\hbar k)^2/2m$
c) plasma oscillation, $\omega^2 = \omega_p^2 + 3k^2 v_{\rm th}^2$.

4. Show that $u \cdot a = 0$, where u is the four-velocity and $a = (d/d\tau)u$ is the acceleration. Then show that $a_0 = \vec{u} \cdot \vec{a}/u_0$.

5. Show that the electric field given in Eq. (8.20) is perpendicular to $\vec{x}$.

6. Consider Eq. (8.24):

(a) Using the fact that $e^2/(\hbar c) = \alpha$ is dimensionless, show that Eq. (8.24) gives dimensions of energy per time.

(b) Suppose you had one Coulomb of charge and dropped it off a building, where it accelerated downward with $g = 9.8$ m/s^2. What power (in W) would be radiated while it fell?

7. The circumference of the LHC is 27 km, and the energy of a proton in the ring is 6.5 TeV. The beam current of the LHC is 0.58 Amperes. (The mass of a proton is 938.3 MeV).

(a) What is the acceleration of a proton? Assume it moves in perfect circular motion, though in reality it passes between magnets for parts of its trajectory, so the acceleration falls between magnets and is higher while the proton is inside the dipoles.

(b) What is the power radiated by the proton?

(c) What fraction of the energy is lost during one revolution of the trajectory?

(d) If electrons were accelerated to the same energy, what would the fraction be? (The mass of an electron is 0.511 MeV)

(e) If electricity can be purchased for 10¢ per kwh, estimate the cost of the LHC due to radiative energy loss for running one day. Give cost for both protons and electrons (if they were put in at the same energy).

8. Consider a particle of charge e moving non-relativistically in a synchotron of radius R with the orbit around the z axis, such that

$$x = R\cos\omega t, \quad y = R\sin\omega t.$$

(a) Find $J_x(\vec{r}, t)$ as defined in Sec.s 8.6 and 8.7.

(b) Mis-stated in original assignment – ignore

(c) Find p_x as defined in Sec. 8.7.

(d) Using Eq. (8.53), what is the radiated power? Be sure to include contribution from both p_x and p_y.

(e) Compare to the result for a non-relativistic point particle moving in a circle from Eq. (8.35).

(f) Why should you not apply Eq. (8.53) in the relativistic case?

9 Scattering

Here we discuss scattering of light, i.e. light comes in with one wave vector, then leaves with another. This includes Thomson scattering, Rayleigh scattering and Compton scattering.

9.1 Scattering in the Long Wavelength Limit

If the object from which light is scattered is small compared to the wavelength of light, it is convenient to think of the problem in two steps. First, the object is excited by the electromagnetic wave, inducing oscillating multipole moments of either the current or charge. These objects then radiate according to the moments. The incoming electromagnetic wave provides a driving force at the frequency of the light, ω. This incoming wave persists for a significant time, which means that the response will settle down to the particular solution of the the differential equation describing the particle with which it interacts. This solution oscillates with the same frequency as the driving frequency. The radiation from this source then also occurs at ω, which means the process can be considered as elastic scattering. The approach we consider here is only valid in the classical limit, i.e. it ignores the momentum and energy of the outgoing particle. For scatterings the momentum transferred to the object, Δp, can approach $2\hbar\omega$ if the scattering is backward and the object is heavy. The outgoing kinetic energy of the object of mass M is then $\Delta p^2/2M \sim \hbar^2\omega^2/2M$, which is negligible for for $\hbar\omega << M$.

9.2 Thomson Scattering

For scattering off a free particle, the $\hbar\omega << M$ limit is known as Thomson scattering. If one accounts for the energy of the outgoing object, the frequency of the scattered light is reduced to conserve the energy of the scattered photon and one has Compton scattering. Thomson scattering was first explained by J.J. Thomson, the same scientist who discovered the electron and established many of the basic precepts of radioactivity. For Thomson scattering, we consider the electric field acting on a particle at the origin. If the wave length is long compared to the subsequent motion, we can consider the electric field as depending on time only,

$$\vec{E}(t) = E_0 \cos\omega t. \tag{9.1}$$

Here, E_0 is the amplitude of the electro-magnetic field. If the polarization is in the x direction, the particle's subsequent motion is given by

$$\begin{aligned} \ddot{x} &= \frac{eE_0}{m}\cos\omega t, \qquad (9.2)\\ x &= -\frac{eE_0}{m\omega^2}\cos\omega t, \end{aligned}$$

where the particle's mass and charge are m and e. The dipole moment and emitted power, see Eq. (8.53), are

$$\begin{aligned} p_x(t) &= -\frac{e^2 E_0}{m\omega^2}\cos\omega t, \qquad (9.3)\\ P &= \frac{\omega^4}{3}p_x^2 \\ &= \frac{e^4 E_0^2}{3m^2}, \end{aligned}$$

which is independent of ω. This can be expressed as a cross section σ by considering the expression for the scattering rate,

$$\Gamma = n\sigma v, \qquad (9.4)$$

where v in this case is the velocity of light, and n is the density of scatterers. The scattering rate Γ is the ratio of power emitted from the scatters within a volume V, divided by the electromagnetic energy of the incoming wave in that volume. The emitted power is the power off a single electron, Eq. (9.3), multiplied by the density of electrons and the overall volume,

$$\begin{aligned} \Gamma &= n\frac{e^4 E_0^2}{3m^2}\left(\frac{8\pi}{E_0^2}\right) \\ &= \frac{8\pi n e^4}{3m^2}, \end{aligned}$$

and the cross section is

$$\sigma = \frac{8\pi e^4}{3m^2}. \qquad (9.5)$$

The differential cross section for polarized light, can be calculated with the help of Eq. (8.52),

$$\begin{aligned} \frac{d\Gamma}{d\Omega} &= \frac{1}{8\pi}k^2|\hat{n}\times\vec{p}|^2\left(\frac{8\pi}{E_0^2}\right) \qquad (9.6)\\ \frac{d\sigma}{d\Omega} &= \frac{e^4}{m^2}(\sin^2\theta\sin^2\phi + \cos^2\theta), \end{aligned}$$

where θ is the angle relative to the incoming wave and ϕ is relative to the polarization. Thus the scattered light prefers to travel transverse to the polarization of the incoming wave.

9.3 Compton Scattering

For Compton scattering, $\hbar$ plays a role and it served as one of the original examples of quantum phenomena in the early 1920s. The effect was first observed by Arthur Compton with X-rays, to make $\hbar\omega$ large, and the scattering was off electrons, the lightest charge particle. In that case energy and momentum conservation required

$$\begin{aligned} \hbar\omega + m &= \hbar\omega' + \sqrt{m^2+p^2}, \qquad (9.7)\\ \hbar\omega &= \hbar\omega'\cos\theta' + p\cos\theta_p, \\ 0 &= \hbar\omega'\sin\theta' + p\sin\theta_p. \end{aligned}$$

Next, we use these equations to find an expression where p and θ_p are eliminated. We first combine the last two equations to obtain

$$p^2 = \hbar^2(\omega^2 + \omega'^2 - 2\omega\omega' \cos\theta'), \tag{9.8}$$

which we then insert into the first equation to obtain,

$$\begin{aligned} (\hbar\omega + m - \hbar\omega')^2 &= m^2 + \hbar^2\omega^2 + \hbar^2\omega'^2 - 2\hbar^2\omega\omega' \cos\theta', \\ 2m\hbar\omega - 2m\hbar\omega' - 2\hbar^2\omega\omega' &= -2\hbar^2\omega\omega' \cos\theta', \\ \frac{1}{\omega} - \frac{1}{\omega'} &= \frac{\hbar(1-\cos\theta')}{m}, \\ \lambda' - \lambda &= \frac{2\pi\hbar(1-\cos\theta')}{m}. \end{aligned} \tag{9.9}$$

where $\lambda = 2\pi/\omega$ is the wave length. This can be written as

$$\frac{\Delta\lambda}{\lambda} = \frac{\hbar\omega}{m}(1-\cos\theta'), \tag{9.10}$$

to see that for low frequencies or very heavy targets, the wave length does not change, and one recovers the Thomson limit.

9.4 Scattering of Light from Confined Charges

Most atoms are neutral, but are made of confined charges. These charges also react to the external electric field. A simple example would be a charge in a damped harmonic oscillator with characteristic frequency ω_0 and damping rate Γ. The equations of motion in an oscillating electric field are

$$\ddot{x} - \Gamma\dot{x} + \omega_0^2 = \left(\frac{eE_0}{m}\right)\cos\omega t. \tag{9.11}$$

The particular solution is of the form, $x = A\cos(\omega t - \phi)$, and the amplitude A determines the dipole moment, which then determines the power from Eq. (8.53), and then the cross section,

$$\begin{aligned} |A|^2 &= \frac{e^2E_0^2/m^2}{(\omega^2-\omega_0^2)^2 + \Gamma^2\omega^2}, \\ p_x &= e|A|\cos(\omega t - \phi), \\ P &= \frac{\omega^4}{3}e^2|A|^2 \\ &= \frac{e^4E_0^2}{3m^2}\frac{\omega^4}{(\omega^2-\omega_0^2)^2 + \Gamma^2\omega^2}, \\ \sigma &= \frac{8\pi e^4}{3m^2}\frac{\omega^4}{(\omega^2-\omega_0^2)^2 + \Gamma^2\omega^2}. \end{aligned} \tag{9.12}$$

If the harmonic oscillator is soft, $\omega_0 << \omega$, one recovers the free particle limit, Thomson scattering. However, if the oscillator is stiff, $\omega_0 >> \omega$, the scattering scales as ω^4. This is the case for visible light, where the energies are one or two eV, but the excitation energies are several eV. For this reason blue light scatters much more than red light, and when one looks at the sky away from the sun, it appears blue. This limit is known as Rayleigh scattering.

9.5 Homework Problems

1. For Thomson scattering, show that for un-polarized light the angular distribution of the scattered light $\sim (1 + \cos^2\theta)$, where θ is the scattering angle.

2. Consider the limit that $\Gamma \to 0$ in Eq. (9.12). When $\omega \to \omega_0$ the cross section then diverges. Does the contribution to the integrated cross section,

$$I(\omega_a, \omega_b) \equiv \int_{\omega_a}^{\omega_b} d\omega\, \sigma(\omega),$$

where ω_a and ω_b confine the integral to the region surrounding ω_0, diverge as well?