PHY 841

HW 8

## 8.8 Homework Problems

1. Consider Eq. (~~8.15~~ 8.14) in the case where $J^\alpha$ has no time dependence. Show that one quickly obtains the usual expressions for the potentials in the static cases.

$$A^\alpha(x) = \int d^4x' \, \frac{1}{|\vec{x} - \vec{x}'|} J^\alpha(x')\delta(x_0 - x_0' - |\vec{x} - \vec{x}'|). \tag{8.15}$$

$$A^\alpha(x) = \int d^4x' \frac{1}{|\vec{x} - \vec{x}'|} J^\alpha(x_0', \vec{x}') \, \delta(x_0 - x_0' - |\vec{x} - \vec{x}'|)$$

If $J^\alpha$ has no time dependence, $\delta$ function integrates to unity

$$A^\alpha(x) = \int d^3x' \frac{1}{|\vec{x} - \vec{x}'|} J^\alpha(\vec{x}')$$

2. Using the fact that $\nabla^2(1/r) = -4\pi\delta^3(\vec{r})$,

(a) show that any function $f(r-t)$ satisfies the differential equation,

$$\partial^2\left(\frac{f(r-t)}{r}\right) = 4\pi f(r-t)\delta^3(\vec{r}).$$

(b) Now, let $f(r-t) = \delta(r-t)$. Show that this satisfies the equation

$$\partial^2\left(\frac{f(r-t)}{r}\right) = 0$$

for all $t > 0$. Also, because $r > 0$ the function is zero for $t < 0$.

(c) Show that the form $f(r-t) = \delta(r-t)$ satisfies the integral of Eq. (8.7).

$$\int_{-\epsilon}^{\epsilon} dt\left[\partial^2\left(\frac{\delta(r-t)}{r}\right)\right] = \cancel{-}4\pi\int dt\,\delta^4(x).$$

a) $\partial^2 = \partial_r^2 + \frac{2}{r}\partial_r$ + angular ...

$$\partial^2\frac{f}{r} = \frac{1}{r}\partial_r^2 f - \cancel{\frac{2}{r^2}\partial_r f} + \cancel{\frac{2}{r^3}f} + \cancel{\frac{2}{r^2}\partial_r f}$$

$$-\cancel{\frac{2}{r^3}f} - \partial_t^2 f = \frac{1}{r}\partial_r^2 f - \frac{\partial_t^2 f}{r}$$

$$\partial^2\left(\frac{f}{r}\right) = \frac{1}{r}\left(\partial_t^2 f - \partial_r^2 f\right)$$

$$f = f(r-t)$$

$$\partial^2\left(\frac{f}{r}\right) = \frac{1}{r}\left(f'' - f''\right) = 0$$

b) From a) $\partial^2 f(r-t) = 0$ for all $t>0$

also $= 0$ for $t = 0$, only non-zero at $t=0$

c) $$\int_{-\epsilon}^{\epsilon} dt\left[\partial^2\frac{\delta(r-t)}{r}\right] = \partial^2\frac{1}{r} = -4\pi\delta^3(\vec{r})$$

$$= -4\pi\int dt\,\delta^4(\vec{r})$$

$$\partial^2\frac{f(r-t)}{r} = -4\pi\,\delta^4(\vec{r})$$

3.)

2. Consider a function $f(x)$ that is a super-position of plane waves,

$$f(x) = \int dk\, g(k') e^{i\omega(k')t - ik'x + i\phi_0(k')},$$

where $g(k')$ is a narrow function centered about $k$, e.g.

$$g(k') = \frac{1}{\sqrt{2\pi a^2}} e^{-(k'-k)^2/2a^2},$$

with $a << k$.

(a) For a given time $t$ find the position $x$ at which the phase $[i\omega(k')t - ik'x + i\phi_0(k')]$ is steady as a function of $k'$ at $k' = k$, i.e.

$$\frac{d}{dk'}[i\omega(k')t - ik'x + i\phi_0] = 0.$$

(b) What are the group velocities for the following cases:
a) massless particle in a vacuum, $\omega = |k|c$
b) massive particles in a vacuum, $\hbar\omega = (\hbar k)^2/2m$
c) plasma oscillation, $\omega^2 = \omega_p^2 + 3k^2 v_{\text{th}}^2$.

a) $\frac{d}{dk'}\left[\omega(k')t - k'x + \varphi_0\right] = 0$

$$\frac{d\omega}{dk} \cdot t - x = 0$$

$$x = \frac{d\omega}{dk} t$$

group vel!

b) a) $v_g = c$

b) $v_g = \hbar k/m$

c) $v_g = \dfrac{3 v_{th} k}{\sqrt{\omega_p^2 + 3k^2 v_{th}^2}}$

4. Show that $\boldsymbol{u} \cdot \boldsymbol{a} = 0$, where $\boldsymbol{u}$ is the four-velocity and $\boldsymbol{a} = (d/d\tau)\boldsymbol{u}$ is the acceleration. Then show that $a_0 = \vec{u} \cdot \vec{a}/u_0$.

$$\frac{d}{dt} u^2 = 0$$

$$= 2u \cdot \frac{du}{d\tau} = 2u \cdot a$$

$$u \cdot a = 0$$

$$u_0 a_0 = \vec{u} \cdot \vec{a}$$

$$a_0 = \vec{u} \cdot \vec{a} / u_0$$

~~4~~5. Show that the electric field given in Eq. (8.21) is perpendicular to $\vec{x}$.

$$\boldsymbol{E}^i = \frac{e}{(\boldsymbol{u}\cdot\boldsymbol{x})^2}\left\{\boldsymbol{x}^0\left(\boldsymbol{a}^i - \frac{\boldsymbol{u}^i(\boldsymbol{a}\cdot\boldsymbol{x})}{(\boldsymbol{u}\cdot\boldsymbol{x})}\right) - \boldsymbol{x}^i\left(\boldsymbol{a}^0 - \frac{\boldsymbol{u}^0(\boldsymbol{a}\cdot\boldsymbol{x})}{(\boldsymbol{u}\cdot\boldsymbol{x})}\right)\right\}.$$

$$\vec{E}\cdot\vec{x} = \frac{e}{(u\cdot x)^2}\left\{ x_0\,\vec{a}\cdot\vec{x} - x_0\frac{\vec{u}\cdot\vec{x}\,(a\cdot x)}{u\cdot x} - |\vec{x}|^2 a_0 + |\vec{x}|^2 u_0\frac{a\cdot x}{u\cdot x}\right\}$$

$$= \frac{e}{(u\cdot x)^3}\left\{ x_0\, u\cdot x\,(\vec{a}\cdot\vec{x}) - x_0(\vec{u}\cdot\vec{x})\, a\cdot x - |\vec{x}|^2 a_0\, u\cdot x + |\vec{x}|^2 u_0 (a\cdot x)\right\}$$

$$|\vec{x}|^2 = x_0^2$$

$$= \frac{e x_0}{(u\cdot x)^3}\left\{ (u\cdot x)(\vec{a}\cdot\vec{x}) - (\vec{u}\cdot\vec{x})\, a\cdot x - x_0 a_0\, u\cdot x + x_0 u_0\, a\cdot x \right\}$$

$$= \frac{e x_0}{(u\cdot x)^3}\left\{ (u\cdot x)(-a\cdot x) + (a\cdot x)(u\cdot x)\right\} = 0$$

6. Consider Eq. (8.24):

(a) Using the fact that $e^2/(\hbar c) = \alpha$ is dimensionless, show that Eq. (8.25) gives dimensions of energy per time.

(b) Suppose you had one Coulomb of charge and dropped it off a building, where it accelerated downward with $g = 9.8\ \mathrm{m/s^2}$. What power (in W) would be radiated while it fell?

$$P = \frac{2e^2}{3c}|\dot{\vec{\beta}}|^2, \tag{8.25}$$

a) $\dfrac{[\hbar c]}{[c]}[\dot{\beta}]^2 = [E\,t]\left[\dfrac{1}{t}\right]^2 = [E/t]$

b) $P = \dfrac{2\hbar c}{3\cdot 137.036} \cdot \dfrac{(g)^2}{c^3} \dfrac{1}{(1.602\cdot 10^{-19})^2}$

$= 2.14 \cdot 10^{-14}$ W

7. The circumference of the LHC is 27 km, and the energy of a proton in the ring is 6.5 TeV. The beam current of the LHC is 0.58 Amperes. (The mass of a proton is 938.3 MeV).

(a) What is the acceleration of a proton? Assume it moves in perfect circular motion, though in reality it passes between magnets for parts of its trajectory, so the acceleration falls between magnets and is higher while the proton is inside the dipoles.

(b) What is the power radiated by the proton?

(c) What fraction of the energy is lost during one revolution of the trajectory?

(d) If electrons were accelerated to the same energy, what would the fraction be? (The mass of an electron is 0.511 MeV)

(e) If electricity can be purchased for 10¢ per kwh, estimate the cost of the LHC due to radiative energy loss for running one day. Give cost for both protons and electrons (if they were put in at the same energy).

a) $a = \frac{v^2}{r} = \frac{c^2}{4297}$ $\quad r = \frac{27000}{2\pi}$

$= 2.09 \cdot 10^{13}$

b) $P = \frac{1}{137.036} \cdot \hbar c \cdot \frac{2}{3} a^2 / c^3 \gamma^4$

$\gamma = 6927$

$P = 5.75 \cdot 10^{-12}$ W

c) $\Delta E = P \cdot 27000 / c$

c) $\Delta E = P \cdot \frac{27000}{c} = 3.23$ keV

$\frac{\Delta E}{E} = 4.97 \cdot 10^{-10}$

d) everything the same except $\gamma$ is higher by $\frac{m_p}{m_e} =$

$\frac{\Delta E}{E} = \left(\frac{m_p}{m_e}\right)^4 \cdot 4.97 \cdot 10^{-10}$

$= 5649$

e) $P = 5.75 \cdot 10^{-15}$ kW

$\#$ electrons $= \frac{0.58 \cdot \frac{27000}{c}}{1.602 \cdot 10^{-19}} = 3.26 \cdot 10^{14}$

e) cont

price $=$

$5.75 \cdot 10^{-15}$

$\cdot 3.26 \cdot 10^{14}$

$\cdot 24 \cdot 0.1$

$= \$4.50$

for electrons

price $=$

$\$4.50 \left(\frac{m_p}{m_e}\right)^4$

$= 51$ trillion dollars

8. Consider a particle of charge $e$ moving non-relativistically in a synchotron of radius $R$ with the orbit around the $z$ axis, such that

$$x = R\cos\omega t, \quad y = R\sin\omega t.$$

(a) Find $J_x(\vec{r}, t)$ as defined in Sec. 8.7.

(b) Find $j_x(\vec{r})$ as defined in Sec. 8.7.

(c) Find $p_x$ as defined in Sec. 8.7.

(d) Using Eq. (8.55), what is the radiated power? Be sure to include contribution from both $p_x$ and $p_y$.

(e) Compare to the result for a non-relativistic point particle moving in a circle from Eq. (8.37).

(f) Why should you not apply Eq. (8.55) in the relativistic case?

a) $J_x = -\omega R\sin\omega t \cdot e\,\delta(z)\,\delta(x - R\cos\omega t) \cdot \delta(y - R\sin\omega t)$

b) SKIP

c) $\rho = J_0 = e\,\delta(z)\,\delta(x-R\cos\omega t)\,\delta(y - R\sin\omega t)$

$$\int J_0\, x\, d^3r = \int j_0(x)\, x\, d^3r\, e^{i\omega t} = p_x e^{i\omega t}$$

$$eR\cos\omega t = p_x e^{i\omega t}$$

$$p_x = eR$$

d) $P = \frac{2\omega^4}{3} e^2 R^2$

$$P = \frac{1}{8\pi} k^2 p^2 \int d\Omega \sin^2\theta = \frac{k^4}{3}|\vec{p}|^2. \quad (8.55)$$

e) For circular motion

$$P = \frac{2}{3} e^2 a^2 = \frac{2}{3} e^2 (\omega^2 R)^2$$

$=$ SAME !

f) time for light to traverse system is not $\gg 2\pi/\omega$.