PHY 841 _ HW 6

solutions

6.5 Homework Problems

1. Prove the following using current conservation ($\nabla \cdot \vec{J} = 0$ for static systems):

   (a) Beginning with

   $$\int d^3r \, (\nabla \cdot \vec{J}(\vec{r}))r_i = 0,$$

   show that

   $$\int d^3r \, J_i(\vec{r}) = 0.$$

   (b) Beginning with

   $$\int d^3r \, (\nabla \cdot \vec{J}(\vec{r}))r_i r_j = 0,$$

   show that p_{ij} is antisymmetric, where

   $$p_{ij} \equiv \int d^3r \, J_i(\vec{r})r_j.$$

a) $$-\int d^3r \, \vec{J} \cdot (\vec{\nabla} r_i) = 0$$

$$-\int d^3r \, \vec{J}_k \, \delta_{ik} = 0$$

$$\int d^3r \, J_i = 0$$

b) $$0 = -\int d^3r \, J_k \, \partial_k (r_i r_j)$$

$$= -\int d^3r \, (J_i r_j + J_j r_i)$$

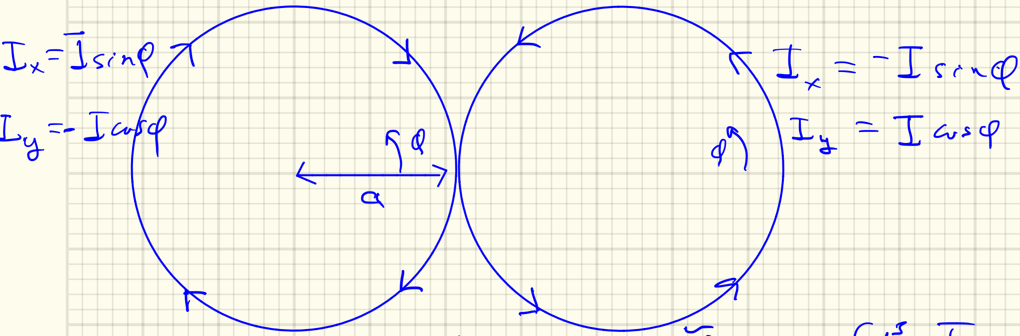
$$= -p_{ij} - p_{ji}$$

2. Consider two current loops, each moving in the ~~$x-y$ plane around the z~~ $z=\emptyset$ along axis. Each loop has current I and radius a. The first loop is centered at $x = -a, y = 0$ and is circulating clockwise, and the second loop is centered at $x =, y = 0$ and is circulating counter-clockwise.

   (a) Calculate the analog of the quadrupole moment for calculating the i^{th} component of the vector potential,

$$Q_{ik\ell} = \int d^3r\, J_i(\vec{r})(3r_k r_\ell - r^2\delta_{k\ell}).$$

   (b) Find the vector potential $A_i(\vec{r})$ far away, to order $1/r^3$. Use relations from the previous chapter, where you merely add an additional index to account for the fact you are dealing with the vector potential. You can leave your answer in terms of $Q_{ik\ell}$, but clarify which components of A_i are zero, and how each component depends explicitly on the angles θ and ϕ.



a) Consider Integral $\tilde{Q}_{ik\ell} \equiv \int d^3r\, J_i r_k r_\ell$

$\int d^3r\, J_i = \int a\, d\varphi\, I_i$

$x = a(\pm 1 + \cos\varphi)$ right/left

$y = a\sin\varphi$

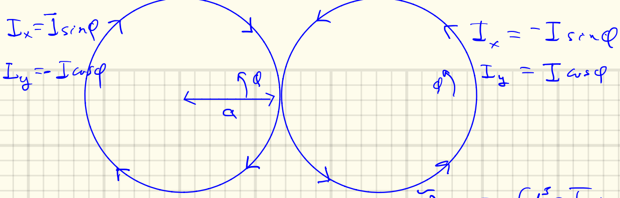
$\tilde{Q}_{zk\ell} = \emptyset$

$\tilde{Q}_{xxx} = \emptyset$ by symmetry

$\tilde{Q}_{xyy} = \emptyset$ by symmetry

$\tilde{Q}_{xxy} = \tilde{Q}_{xyx} = 2a^3 \int d\varphi\, (1+\cos\varphi)\sin\varphi\,(-\sin\varphi)\, I$

$= -2\pi a^3 I$



$\tilde{Q}_{ikl} \equiv \int d^3r\, J_i r_k r_l$

$\int d^3r\, J_i = \int a\, d\varphi\, I_i$

$x = a(\pm 1 + \cos\varphi)$ right/left

$y = a\sin\varphi$

a) cont

$$\tilde{Q}_{yxx} = 2a^3 \int d\varphi\, I\cos\varphi\,(1+\cos\varphi)^2 = 4\pi a^3 I$$

$\tilde{Q}_{yyy} = \emptyset$ by symmetry

$\tilde{Q}_{yxy} = \tilde{Q}_{yyx} = \emptyset$ by symmetry

$$\tilde{Q}_{irr} \equiv \int d^3r\, J_i r^2$$

$\tilde{Q}_{xrr} = \emptyset$ by symmetry

$$\tilde{Q}_{yrr} = 2a^3 I \int d\varphi\, I\cos\varphi\,[2+2\cos\varphi] = 4\pi a^3 I$$

$$Q_{xij} = \begin{pmatrix} 0 & -3 & 0 \\ -3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot 2\pi a^3 I$$

$$Q_{yij} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix} \cdot 2\pi a^3 I$$

$$Q_{zij} = \emptyset$$

b)

$$q_{\ell m} = \int d^3r' \, r'^{\ell} \rho(\vec{r}') Y^*_{\ell m}(\theta', \phi').$$

$$q_{00} = \frac{1}{\sqrt{4\pi}} \int d^3r \, \rho(\vec{r}) = \frac{1}{\sqrt{4\pi}} q,$$

$$q_{11} = -\sqrt{\frac{3}{8\pi}} \int d^3r \, \rho(\vec{r})(x - iy) = -\sqrt{\frac{3}{8\pi}}(p_x - ip_y),$$

$$q_{10} = -\sqrt{\frac{3}{4\pi}} \int d^3r \, \rho(\vec{r}) z = -\sqrt{\frac{3}{4\pi}} p_z,$$

$$q_{22} = \sqrt{\frac{15}{32\pi}} \int d^3r \, \rho(\vec{r})(x - iy)^2 = \sqrt{\frac{15}{288\pi}}(Q_{11} - 2iQ_{12} - Q_{22}),$$

$$q_{21} = -\sqrt{\frac{15}{8\pi}} \int d^3r \, \rho(\vec{r})(x - iy)^2 = -\sqrt{\frac{15}{72\pi}}(Q_{13} - iQ_{23}),$$

$$q_{20} = \sqrt{\frac{5}{16\pi}} \int d^3r \, \rho(\vec{r})(3z^2 - r^2) = \sqrt{\frac{5}{16\pi}} Q_{33}.$$

Here q is the net charge

$$\rho \to J_i$$

$$\Phi(\vec{r}) = \sum_{\ell m} \frac{4\pi}{(2\ell + 1)} q_{\ell m} \frac{Y_{\ell m}(\theta, \phi)}{r^{\ell+1}}.$$

$$\Phi \to A_i$$

$$A_x = \sum_{\ell, m} \frac{4\pi}{(2\ell+1)} q_{\ell m} \frac{Y_{\ell m}(\theta, \varphi)}{r^{\ell+1}}$$

$$q_{22} = \sqrt{\frac{15}{288\pi}} (-2i) Q_{xxy}, \quad q_{2-2} = q_{22}^*$$

$$A_x = \frac{4\pi}{5r^3}(-2i)\sqrt{\frac{15}{288\pi}} \cdot 2\pi a^3 I \, Y_{22} + c.c.$$

$$Y_{22} = \sqrt{\frac{15}{32\pi}} \sin^2\theta \, e^{2i\varphi}$$

$$= \frac{\sin^2\theta}{r^3} a^3 I \sin 2\varphi \left[\frac{4\pi}{5} \cdot 2\sqrt{\frac{15 \cdot 15}{288 \cdot 32}}\right] \frac{1}{\pi} \cdot 2\pi \cdot 2$$

$$= \frac{\sin^2\theta \sin 2\varphi \, a^3 I}{r^3} \pi \cdot \frac{24 \cdot 2}{12 \cdot 8} \cdot 2$$

$$= \frac{\sin^2\theta \sin 2\varphi}{r^3} \pi a^3 I$$

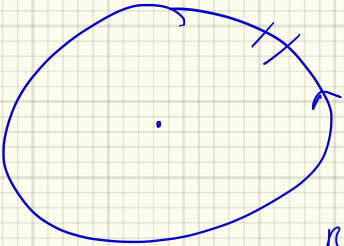
For A_y, $q_{22} = \sqrt{15/288\pi} \cdot 12\pi a^3 I$

$$q_{20} = \sqrt{5/16\pi} \; -4\pi a^3 I$$

$$A_y = \frac{\sin^2\theta \cos 2\varphi}{r^3} \pi a^3 I + \frac{4\pi}{5r^3} \sqrt{\frac{5}{16\pi}} \, 4\pi a^3 I \, Y_{20}, \quad Y_{20} = \sqrt{\frac{5}{16\pi}}(3\cos^2\theta - 1)$$

$$= \frac{\sin^2\theta \cos 2\varphi}{r^3} \pi a^3 I - \frac{(3\cos^2\theta - 1)}{r^3} \pi a^3 I$$

3. An particle of mass m and charge e moves in a circular orbit of radius R with angular momentum $\boldsymbol{L}$. What is the magnitude of the magnetic field at the origin? Note this was used in writing the expression in Eq. (6.34) for the hyper-fine coupling.



$$dB = \frac{\vec{J}\, d^3r \times \vec{r}}{r^3}$$

$$B = \int \frac{I\, d\varphi}{r} = \frac{2\pi I}{R}$$

$$I = \frac{e}{\tau} = \frac{e}{2\pi R/v} = \frac{ev}{2\pi R}$$

$$B = \frac{2\pi}{R} \cdot \frac{ev}{2\pi R} = \frac{ev}{R^2}$$

$$= \frac{e\, mvR}{R^3}$$

4. Using Eq. (6.34), consider the hyper-fine energies for the 1s levels of a hydrogen atom. There are two levels because the electron and proton spin can couple to either zero or unity.

   (a) Which terms in Eq. (6.34) contribute to the energy?

   (b) Find the difference between the energy levels. Express your answer numerically in eV, using the electron mass and charge, and using the fact that the hydrogen atom electron density for the 1s state behaves as

$$\rho(\vec{r}) \sim e^{-2r/a_0},$$

   $a_0 = 0.529$ Å.

   (c) What would the wavelength of light be for a transition between the states.

$$U = \frac{3(\vec{\mu}_N \cdot \vec{r})(\vec{\mu}_e \cdot \vec{r})}{r^5} - \frac{(\vec{\mu}_N \cdot \vec{\mu}_e)}{r^3} - \frac{8\pi}{3}(\vec{\mu}_N \cdot \vec{\mu}_e)\delta^3(\vec{r}) - e\frac{(\vec{\mu}_N \cdot \vec{L})}{mr^3}. \quad (6.34)$$

a) s-wave, so first 2 terms cancel each other, last term also disappears

$$U = -\frac{8\pi}{3}(\vec{\mu}_N \cdot \vec{\mu}_e)\,\delta^3(\vec{r})$$

b) $\vec{\mu}_N \cdot \vec{\mu}_e = g_N g_e \cdot \vec{S}_N \cdot \vec{S}_e = g_N g_e \left(\frac{e\hbar}{2m_e}\,\frac{e\hbar}{2m_N}\right)$

$$\cdot \left(\frac{\vec{S}_N}{\hbar}\cdot\frac{\vec{S}_e}{\hbar}\right) = g_N g_e \frac{e^2\hbar^2}{4m_e m_N} \cdot \frac{1}{2}\Big(S(S+1) - S_N(S_N+1) - S_e(S_e+1)\Big)$$

$$\frac{1}{2}\left(2-\frac{3}{2}\right) = \frac{1}{4}$$

$$\sim -3/4$$

$$= g_N g_e \frac{e^2\hbar^2}{4m_e^2 m_N^2}\left(\frac{1}{4} \text{ for } S=1 \quad -\frac{3}{4} \text{ for } S=0\right)$$

$$\Delta U = g_N g_e \frac{e^2\hbar^2}{4m_e^2 m_N^2} \cdot \rho(r=0) \cdot \frac{8\pi}{3}$$

$$\rho(\vec{r}) = \rho_0 e^{-2r/a_0}$$

$$4\pi \rho_0 \int r^2 dr\, e^{-2r/a_0} = 1$$

$$\rho_0 = \frac{1}{4\pi \cdot 2 (a_0/2)^3} = \frac{1}{\pi a_0^3}$$

$$\Delta U = g_N g_e \frac{e^2 \hbar^2}{4 m_N m_e c^2} \frac{8\pi}{3} \frac{1}{\pi a_0^3}$$

$$= \frac{2}{3} g_N g_e \frac{e^2 \hbar^2}{m_N m_e} \frac{1}{a_0^3}$$

$$e^2 = \hbar c \alpha$$

$$\alpha = 1/137.036$$

$$g_e = 2$$

$$g_p = 5.58$$

$$= \frac{2}{3} g_N g_e \frac{\hbar^3 \alpha c}{m_N m_e c^2} \frac{1}{a_0^3}$$

$$= 5.86 \, 10^{-6}\, eV$$

$$\lambda = 21\ cm$$

5. Consider a parallel-plate capacitor where the area of the plates is A and the small separation is a. The charge on the plates are $\pm Q$.

   (a) What is the dipole moment of the capacitor if the plates are aligned in the z direction?

   (b) What is the electric field in the capacitor?

   (c) Show that Eq. (6.34) is satisfied in this case where the spherical volume engulfs the entire capacitor.

a) $p_z = Qa$

b) $\sigma \cdot A = EA/4\pi$

$E_z = -4\pi\sigma = -4\pi Q/A$

c) $\int d^3r\, E_z = -4\pi Q a = -4\pi p_z$

NOT THE SAME!

off by $\times 3$!!