PHY 841

Chapter 3
HW Solutions

3.5 Homework Problems

1. Consider a sphere of radius R and charge Q, where the charge is spread uniformly throughout the sphere.

   (a) Find the strength of the electric field as a function of r.

   (b) Find the electric potential as a function of r.

   (c) Find the potential energy required to move the charges to their positions,

   $$PE = \frac{1}{2}\int d^3r\, \rho(r) V(r).$$

   (d) Find the energy contained in the electric fields.

a) Gauss's Law

$$4\pi Q \frac{r^3}{R^3} = E \cdot 4\pi r^2$$

$$E = \frac{Qr}{R^3}, \quad r < R$$

$$= \frac{Q}{r^2}, \quad r > 0$$

b)

$$\Phi = \int_r^\infty dr\, E$$

$$= \begin{cases} \dfrac{Q}{r}, & r > R \\ -\dfrac{Qr^2}{2R^3} + \dfrac{QR^2}{2R^3} + \dfrac{Q}{R}, & r < R \end{cases}$$

$$= \begin{cases} Q/R, & r > R \\ \dfrac{3Q}{R} - \dfrac{Qr^2}{2R^3}, & r < R \end{cases}$$

c) $$PE = \frac{1}{2}\,\frac{3Q^2}{4\pi R^3}\int 4\pi r^2 dr\left\{\frac{3}{2R} - \frac{r^2}{2R^3}\right\}$$

$$= \frac{3Q^2}{2R}\left\{\frac{3}{2}\cdot\frac{1}{3} - \frac{1}{2}\cdot\frac{1}{5}\right\}$$

$$= \frac{3}{5}\,\frac{Q^2}{R}$$

d) $$U = \frac{Q^2}{8\pi}\int_0^R 4\pi r^2 dr\left(\frac{r^2}{R^6}\right)$$

$$+ \frac{Q^2}{8\pi}\int_R^\infty 4\pi r^2 dr\,\frac{1}{r^4}$$

$$= \frac{Q^2}{2R}\left\{\frac{1}{5} + 1\right\} = \frac{3}{5}\,\frac{Q^2}{R} \quad \checkmark$$

2. Beginning with $F^{\alpha\beta}$ written in term of $\vec{E}$ and $\vec{B}$, restate

$$\begin{aligned}\partial_\alpha F^{\alpha\beta} &= 4\pi j^\beta\\ \partial_\alpha \tilde{F}^{\alpha\beta} &= 0\end{aligned}$$

as

$$\begin{aligned}\nabla\cdot\vec{E} &= 4\pi\rho,\\ \nabla\times\vec{B} &= \partial_t\vec{E}+4\pi\vec{j},\\ \nabla\cdot\vec{B} &= 0,\\ \nabla\times\vec{E} &= -\partial_t\vec{B}.\end{aligned}$$

$F^{0i} = -E^i$, $F^{i0} = +E^i$, $F^{\alpha\alpha} = 0$

$F^{ij} = -\varepsilon_{ijk} B^k$

(1) $\cancel{\partial_t F^{00}} + \partial_i F^{i0} = 4\pi j^0$

$\vec{\nabla}\cdot\vec{E} = 4\pi j^0$ ✓

(2) $\partial_t F^{0j} + \partial_i F^{ij} = 4\pi j^j$

$-\partial_t E^i - \partial_i \varepsilon_{ijk} B_k = 4\pi j^j$

$-\partial_t E^i + \varepsilon_{jik}\partial_i B_k = 4\pi j^i$

$-\partial_t \vec{E} + \vec{\nabla}\times\vec{B} = 4\pi\vec{J}$ ✓

$\tilde{F}^{0i} = -B_i$, $\tilde{F}^{i0} = +B_i$

$\tilde{F}^{ij} = \varepsilon_{ijk} E_k$

(3) $\partial_t \tilde{F}^{00} + \partial_i \tilde{F}^{i0} = \vec{\nabla}\cdot\vec{B} = 0$ ✓

(4) $\partial_t \tilde{F}^{0j} + \partial_i \tilde{F}^{ij} = -\partial_t B_j + \partial_i \varepsilon_{ijk} E_k$

$= -\partial_t B_j - \varepsilon_{jik}\partial_i E_k$, $\dfrac{\partial\vec{B}}{\partial t} = -\vec{\nabla}\times\vec{E}$ ✓

3. For a charge-free region, $j^\alpha = 0$

(a) Use Maxwell's equations to write a wave equation for $\vec{E}$, and show the speed of propagation is unity (c).

(b) For a wave traveling in the $\hat{z}$ direction with the electric field in the $\pm\hat{x}$ direction, write a solution for the propagating plane wave for both $\vec{E}(\vec{r}, t)$ and $\vec{B}(\vec{r}, t)$.

a) $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

$\vec{\nabla} \times \vec{B} = \frac{\partial \vec{E}}{\partial t}$

$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -\frac{\partial}{\partial t} \vec{\nabla} \times \vec{B} = -\frac{\partial^2}{\partial t^2} \vec{E}$

$-\nabla^2 \vec{E} + \nabla(\cancel{\vec{\nabla} \cdot E}) = -\partial_t^2 \vec{E}$,

$E \sim e^{ikz - i\omega t}$, $\omega = k$, so speed $= \frac{d\omega}{dk} = 1$

b) Let $\vec{E} = E_0 \hat{x} e^{ikz} e^{-i\omega t}$, $\omega = k$

$\vec{B} = E_0 \hat{y} e^{ikz} e^{-i\omega t}$

$\vec{\nabla} \times \vec{E} = ik E_0 \hat{y} e^{ikz - i\omega t}$

$-\frac{\partial \vec{B}}{\partial t} = i\omega E_0 \hat{y} e^{ikz - i\omega t}$

check!

$\vec{\nabla} \times \vec{B} = -ik E_0 \hat{x} e^{ikz - i\omega t}$

$\frac{\partial \vec{E}}{\partial t} = -i\omega E_0 \hat{x} e^{ikz - i\omega t}$

check

4. First calculate $\hbar c$ in standard mks units. Then, using the fact that the charge on an electron is 1.602×10^{-19} Coulombs, find the constant k in mks units used in Coulomb's law, $PE = kq^2/r$. Use the fact that $PE = e^2/r$, where $e^2 = \hbar c/137.036$.

$$PE = \frac{k \cdot (1.602 \cdot 10^{-19})^2}{r}$$

$$= \frac{\hbar c}{137.036} \frac{1}{r}$$

$$k = \frac{\hbar c}{137.036} \cdot \frac{1}{(1.602 \cdot 10^{-19})^2}$$

$$= \frac{6.634 \cdot 10^{-34}\ Js \cdot 2.998 \cdot 10^{8}\ \frac{m}{s}}{137.036 \cdot (1.602 \cdot 10^{-19})^2\ C^2}$$

$$= 8.99 \cdot 10^{9}\ \frac{Jm}{C^2}$$

5. Consider two very large parallel capacitor plates of area A, carrying charge densities σ and $-\sigma$, and oriented perpendicular to the z axis. The plates are initially at a very small separation at $t = 0$, but are pulled apart, moving with constant non-relativistic velocities $v/2$ and $-v/2$.

(a) What is the electric field between the plates?

(b) Find all four non-zero elements of the stress-energy tensor (T_{xx}, T_{yy}, T_{zz} and T_{00}). Check that the stress-energy tensor is traceless.

(c) In hydrodynamics, the work done by an expanding a gas is PdV. Here, because the expansion is along the z axis the work is $T_{zz}dV$. What is the power required to pull the plates apart at these velocities?

(d) What is energy density of the field between the plates?

(e) What is the rate (energy per time) at which the field energy between the plates increases due to the growing volume?

a) $E \cdot A = 4\pi\sigma \cdot A \quad , \quad E_z = 4\pi\sigma$

b) $T^{00} = \frac{E^2}{8\pi} \quad , \quad T^{xx} = \frac{-E^2}{8\pi} \quad , \quad T^{yy} = T^{zz} = +\frac{E^2}{8\pi}$

$T^\alpha{}_\alpha = \frac{E^2}{8\pi}(1 + 1 - 2) = \emptyset$

c) $T_{\cancel{zz}} \cdot A v = \text{Force} \cdot v$

$= \frac{E^2}{8\pi} \cdot v$

d) $\frac{1}{8\pi} E^2$

e) $\frac{d}{dt}\left(\frac{E^2}{8\pi} \cdot \underset{u}{\underset{\parallel}{A}} \cdot vt\right) = \frac{E^2}{8\pi} V$