PRACTICE

When you are not practicing, remember, someone somewhere is practicing, and when you meet him he will win – E. Macauley

---

THINGS TO MEMORIZE

$$S = -\sum_{\ell} p_\ell \ln(p_\ell)$$

$$TdS = dE + PdV - \mu dN, \quad TS = E + PV - \mu N.$$

$$dN_{\text{free 1-body states}} = (2J+1)\frac{d^dp d^dr}{(2\pi\hbar)^d} = (2J+1)\frac{d^dk d^dr}{(2\pi)^d}$$

$$f(\mathbf{p})_{\text{bosons/fermions}} = \frac{e^{-\beta(\epsilon-\mu)}}{1 \mp e^{-\beta(\epsilon-\mu)}}$$

$$C_V \equiv T\left.\frac{dS}{dT}\right|_{V,N}, \quad C_P \equiv T\left.\frac{dS}{dT}\right|_{P,N}$$

$$\langle q\frac{\partial H}{\partial q}\rangle = T, \quad \langle p\frac{\partial H}{\partial p}\rangle = T, \quad \langle q\dot{p}\rangle = -\langle p\dot{q}\rangle.$$

Don't memorize factors, but know dependencies of more complicated expressions. For instance, you wouldn't be expected to memorize all the $2^{1/2}$ factors in:

$$P = \rho T\left[1 + \sum_{n=2}^{\infty} A_n\left(\frac{\rho}{\rho_0}\right)^{n-1}\right], \quad \rho_0 \equiv B_1 = \frac{(2j+1)}{(2\pi\hbar)^3}\int d^3p\; e^{-\epsilon_p/T} = \frac{(mT)^{3/2}}{(2\pi\hbar^2)^{3/2}}.$$

$$\Delta\frac{dN}{d\epsilon} = \frac{1}{\pi}\sum_{\ell}(2\ell+1)\frac{d\delta_\ell}{d\epsilon}, \quad A_2 = -2^{3/2}\sum_{\ell}\int d\epsilon\; \frac{(2\ell+1)}{\pi}\frac{d\delta}{d\epsilon}e^{-\epsilon/T}.$$

$$\delta = -ap/\hbar, \quad (a = \text{scattering length}),$$

but you would be expected to know what happens to the second virial coefficient if a repulsive or attractive interaction is added, or in what way the pressure would change if a resonance was included.

---

1. Consider two neutrons (spin 1/2 particles). They occupy a two-level system where the single-particle energies are $-\epsilon$ and ϵ, which is thermalized at a temperature T.

   (a) What is the average energy $\langle E\rangle$ of the system as a function of T?

   (b) What is the chance that the ground state is occupied as a function of T?

   (c) What is the entropy S of the system as a function of T?

2. Beginning with the expression,

$$TdS = dE + PdV - \mu dN$$

derive the Maxwell relations

$$\frac{1}{T^2}\left.\frac{dT}{d(\beta\mu)}\right|_E = -\left.\frac{dN}{dE}\right|_{\mu/T}.$$

and

$$-\rho^2\left.\frac{d(S/N)}{d\rho}\right|_T = \left.\frac{dP}{dT}\right|_\rho$$

3. Using the last Maxwell relation from the previous problem, show that if P and S/N are functions of T and ρ, that

$$\left.\frac{dP}{d\rho}\right|_{S/N} = \left.\frac{\partial P}{\partial \rho}\right|_T + \left(\left.\frac{\partial P}{\partial T}\right|_\rho\right)^2 \frac{T}{\rho^2 C_V}.$$

4. Consider a system with an order parameter denoted by x along with a particle number N, energy E and volume V. The system will maximize entropy if

$$\left.\frac{dS}{dx}\right|_{E,V,N} = 0.$$

Show that if the system has a fixed volume and is connected to a bath at temperature T_B and chemical potential μ_B that can can exchange both particles and energy, that the total entropy (of both the system and the bath) will be maximized if the pressure is maximized, i.e.,

$$\left.\frac{dP}{dx}\right|_{T=T_B,\mu=\mu_B,V} = 0.$$

You may use the identity,

$$PV = TS - E + \mu N$$

5. Consider a particle moving in one-dimension according to the Hamiltionian

$$H = \sqrt{p^2 + m^2} + Bx^4$$

Using the equipartion, generalized equi-partition or virial theorems, find

$$\langle x^4 \rangle.$$

6. Consider a thermalized two-dimensional gas of charged non-interacting massless spin-zero bosons, whose energies are given by:

$$\epsilon = pc.$$

Find the density (number per area) required for Bose condensation. Give answer in terms of c, T and $\hbar$.

7. Consider the one-dimensional case for the massless particles from the previous problem. Can you solve for the critical density (number per length) in this case? For massless particles, what is the minimum number of dimensions for Bose condensation, and how does this compare with the massive case?

8. Consider a two-dimensional gas of non-interacting non-relativistic electrons of mass m.

   (a) What is the density of single-particle states $D(\epsilon)$ at the Fermi surface? (in terms of ϵ_f and m)

   (b) What is the energy per area for $T = 0$? (in terms of m and ϵ_f)

   (c) In terms of $D(\epsilon_f)$ and derivatives of $D(\epsilon_f)$, find the lowest-order (expanding in T) non-zero correction to the energy density at low temperature.

9. Consider the equation of state,

$$P(N, T, V) = \frac{NT}{V - Nv_0} - a\left(\frac{N}{V}\right)^2.$$

   Solve for the critical point T_c and ρ_c in terms of v_0 and a.

10. Consider a two-dimensional lattice of three-dimensional oscillators, where the speed of sound for both longitudinal and transverse modes is c_s.

    (a) What is the Debye frequency? Give ω_D in terms of c_s and N/A, the number of lattice sites per area.

    (b) What is the energy of a system of N sites as a function of T if $T >> \hbar\omega_D$?

11. Consider the mean field approximation to the Ising model where the Hamiltonian is

$$H = \sum_i -\mu B \sigma_i - qJ\langle\sigma\rangle\sigma_i,$$

    where σ_i takes on values of -1 or 1.

    (a) Beginning with the definition of $\langle\sigma\rangle$,

$$\langle\sigma\rangle = \frac{\sum_{\sigma_i} \sigma_i e^{-\beta H(\sigma_i)}}{\sum_{\sigma_i} e^{-\beta H(\sigma_i)}},$$

    derive a transcendental expression for $\langle\sigma\rangle$ as a function of μB and qJ.

    (b) What is the critical temperature T_c?

    (c) Find the susceptibility, $\chi = d\langle\sigma\rangle/dB$, for $T >> T_c$? Give answer to lowest non-zero order in $1/T$.

12. Beginning with the partition function,

$$Z(T, \mu B) = \mathrm{Tr}\ e^{-\beta(H_{\rm int} - \int d^3r\ \mu\vec{B}\cdot\vec{\sigma})},$$

where $H_{\rm int}$ is the internal Hamiltonian (that part not involving the external fields), derive and expression for the integrated spin-spin correlation function,

$$\gamma \equiv \int d^3r \left\{ \langle \sigma(r=0)\sigma(r) \rangle - \langle \sigma \rangle^2 \right\},$$

in terms of $f \equiv T \ln Z/V$ and derivatives of f.

13. Beginning with the expression:

$$\langle \sigma \rangle = \tanh(\beta J \langle \sigma \rangle + \beta \mu B)),$$

(a) Derive the value of the critical exponent β defined by

$$\langle \sigma \rangle|_{B=0, T<T_c} \sim |t|^\beta, \text{for } t \equiv (T - T_c)/T_c.$$

(b) Derive the value of the critical exponent δ defined by

$$\langle \sigma \rangle|_{T=T_c} \sim B^{1/\delta}$$

14. In the $X - Y$ model a spin can point in any direction θ in the two-dimensional $x - y$ plane. Assuming neighboring spins in a two-dimensional square lattice interact with a potential energy,

$$V_{ij} = V_0[1 - \cos(\theta_i - \theta_j)],$$

derive the coefficient κ used in Landau theory, where the free energy density (free energy per unit area) has a term

$$f = \frac{1}{2}\kappa(\nabla\theta)^2.$$

Express κ in terms of V_0 and the lattice spacing a.