Chapter 5.2-5.3 Review Solution

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1 Problem 1

Given $\langle\sigma\rangle = \tanh\left(\beta qJ\langle\sigma\rangle\right)$, find an expression for $\langle\sigma\rangle$ in the limit that $\delta T = T_c - T$ is small. Assume that there is no external magnetic field.

We are given that

$$\langle\sigma\rangle = \tanh\left(\beta qJ\langle\sigma\rangle\right) \tag{1}$$

We need to expand the hyperbolic tangent function:

$$\tanh(x) = x - \frac{1}{3}x^3 + ...$$

$$\langle\sigma\rangle = \beta qJ\langle\sigma\rangle - \frac{1}{3}(\beta qJ)^3\langle\sigma\rangle^3 + ...$$

Solving this for $\langle\sigma\rangle$, we find

$$\langle\sigma\rangle \approx \sqrt{\frac{3(\beta qJ - 1)}{(\beta qJ)^3}} \tag{2}$$

Using the fact that $T_c = qJ$ and that the difference between T and T_c is small, we obtain the result after some algebra:

$$\langle\sigma\rangle = \sqrt{\frac{3\left(\frac{T_c}{T} - 1\right)}{\left(\frac{T_c}{T}\right)^3}}$$

$$\langle\sigma\rangle = \sqrt{\frac{3(T_c - T)}{\frac{T_c^3}{T^2}}}$$

$$\boxed{\langle\sigma\rangle = \sqrt{\frac{3(T_c - T)}{T_c}}} \tag{3}$$

2 Problem 2

Consider a one-dimensional Ising model at temperature $T > 0$. Answer yes or no to the following questions:

2.1 Part a

Is spontaneous magnetization possible?

No, a one-dimensional system cannot undergo spontaneous magnetization. For any temperature above zero, it is possible that any two consecutive spins will not point in the same direction.

2.2 Part b

Is the statement "There is no long range order unless there is long range interaction" true?

Yes, the Ising model describes short range interactions, so this statement is true. 1d has no phase transition due to the infinite energy required to break the line into two segments identical except for spin.

2.3 Part c

Is there phase transition in the exact solution?

No, the one-dimensional Ising model has no phase transition. Phase transitions are associated with a nonanalytic behavior of the free energy in the thermodynamic limit. The one-dimensional solution is nonanalytic at $T = 0$, but for $T > 0$, it is analytic, so it does not allow phase transitions.