PHY 831 HW Solutions

Contents

1 Chapter 1

1.12 Problems

1. Using the methods of Lagrange multipliers, find x and y that minimize the following function,

$$f(x, y) = 3x^2 - 4xy + y^2,$$

subject to the constraint,

$$3x + y = 0.$$

Minimize $g = 3x^2 - 4xy + y^2 - \lambda(3x + y)$

$\frac{\partial g}{\partial x} = 0 = 6x - 4y - 3\lambda$ (1)

$\frac{\partial g}{\partial y} = 0 = -4x + 2y - \lambda$ (2)

$\frac{\partial g}{\partial \lambda} = 0 = 3x + y$ (3)

From (1) & (2)

$0 = 18x - 10y = 0$

inserting (3)

$0 = 18x + 30x = 0$, $x = \phi$, $y = \phi$

2. Consider 2 identical bosons (A given level can have an arbitrary number of particles) in a 2-level system, where the energies are 0 and ϵ. In terms of ϵ and the temperature T, calculate:

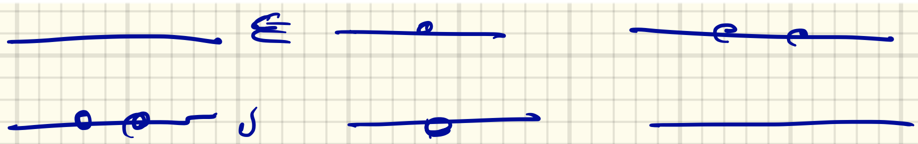
(a) The partition function Z_C

(b) The average energy $\langle E \rangle$. Also, give $\langle E \rangle$ in the $T = 0, \infty$ limits.

(c) The entropy S. Also give S in the $T = 0, \infty$ limits.

(d) Now, connect the system to a particle bath with chemical potential μ. Calculate $Z_{GC}(\mu, T)$. Find the average number of particles, $\langle N \rangle$ as a function of μ and T. Also, give the $T = 0, \infty$ limits.
Hint: For a grand-canonical partition function of non-interacting particles, one can state that $Z_{GC} = Z_1 Z_2 \cdots Z_n$, where Z_i is the partition function for one single-particle level, $Z_i = 1 + e^{-\beta(\epsilon_i - \mu)} + e^{-2\beta(\epsilon_i-\mu)} + e^{-3\beta(\epsilon_i-\mu)} \cdots$, where each term refers to a specific number of bosons in that level.

STATES



a) $Z = e^{2\varepsilon/t} + 1 + e^{-2\varepsilon/t}$

b) $E = \dfrac{-2\varepsilon e^{2\varepsilon/t} + 2\varepsilon e^{-2\varepsilon/t}}{Z}$

c) $S = \ln Z + E/T$

$S(T\to 0) = \emptyset$

$S(T\to\infty) = \ln 3$

d) $Z = Z_0 \cdot Z_\varepsilon$

$Z_0 = 1 + e^{\beta\mu} + e^{2\beta\mu} + \cdots e^{n\beta\mu} = \dfrac{1}{1 - e^{+\beta\mu}}$

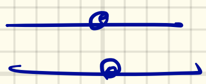
$Z_\varepsilon = \dfrac{1}{1 + e^{-\beta(\varepsilon-\mu)}}$

$\langle N \rangle = \dfrac{\partial \ln Z}{\partial \beta\mu} = \dfrac{e^{\beta\mu}}{1 - e^{\beta\mu}} + \dfrac{e^{-\beta(\varepsilon-\mu)}}{1 - e^{-\beta(\varepsilon-\mu)}}$, $\mu < 0$

$\langle N \rangle_{T\to 0} = \emptyset$

$\langle N \rangle_{T=\infty} = -\dfrac{T}{\mu} - \dfrac{T}{(\mu-\varepsilon)}$, $\mu < 0$!

3. Repeat the problem above assuming the particles are identical Fermions (No level can have more than one particle, e.g., both are spin-up electrons).



only one state

a) $Z = e^{-\beta \varepsilon}$

b) $E = \varepsilon$

c) $S = \emptyset$

d) $\mathcal{Z} = \mathcal{Z}_0 \mathcal{Z}_\varepsilon$, $\mathcal{Z}_0 = 1 + e^{\beta\mu}$, $\mathcal{Z}_\varepsilon = (1 + e^{-\beta(\varepsilon-\mu)})$

$$\langle N \rangle = \frac{e^{\beta\mu}}{1+e^{\beta\mu}} + \frac{e^{-\beta(\varepsilon-\mu)}}{1+e^{-\beta(\varepsilon-\mu)}}$$

$$\langle N \rangle_{T\to 0} = \Theta(\mu) + \Theta(\mu - \varepsilon)$$

4. Beginning with the expression,

$$TdS = dE + PdV - \mu dQ,$$

show that the pressure can be derived from the Helmholtz free energy, $F = E - TS$, with

$$P = -\left.\frac{\partial F}{\partial V}\right|_{Q,T}.$$

$$dF = d(E - TS) = -SdT - PdV + \mu dQ$$

$$P = -\left.\frac{\partial F}{\partial V}\right|_{T,Q}$$

5. Assuming that the pressure P is independent of V when written as a function of μ and T, i.e., $\ln Z_{GC} = PV/T$ (true if the system is much larger than the range of interaction),

   (a) Find expressions for E/V and Q/V in terms of P, T, and partial derivatives of P or P/T w.r.t. $\alpha \equiv -\mu/T$ and $\beta \equiv 1/T$. Here, assume the chemical potential is associated with the conserved number Q.

   (b) Find an expression for $C_V = dE/dT|_{Q,V}$ in terms of $P/T, E, Q, V$ and the derivatives of $P, P/T, E$ and Q w.r.t. β and α.

   (c) Show that the entropy density $s = \partial_T P|_\mu$.

$$PV/T = \ln Z \,, \quad Z = \mathrm{Tr}\, e^{-\beta(\varepsilon - \mu)}$$

a)
$$\langle E \rangle = -\frac{\partial \ln Z}{\partial \beta} = -V\frac{\partial(\beta P)}{\partial \beta}$$

$$\langle Q \rangle = -\frac{\partial \ln Z}{\partial \alpha} = -V\frac{\partial(\beta P)}{\partial \alpha}$$

b)
$$C_V = -\beta^2 \frac{\partial E}{\partial \beta}\Big|_Q$$

$$\delta\langle E \rangle = \frac{\partial E}{\partial \beta}\delta\beta + \frac{\partial E}{\partial \alpha}\delta\alpha$$

$$\delta\langle Q \rangle = \frac{\partial Q}{\partial \beta}\delta\beta + \frac{\partial Q}{\partial \alpha}\delta\alpha = 0$$

$$\delta\langle E \rangle|_\alpha = \delta\beta\left\{\frac{\partial E}{\partial \beta} - \frac{\partial E}{\partial \alpha}\frac{\partial Q}{\partial \beta}\frac{1}{\frac{\partial Q}{\partial \alpha}}\right\}$$

$$C_V = \beta^2\left\{\frac{\partial_\alpha E\,\partial_\beta Q}{\partial_\alpha Q} - \partial_\beta E\right\}$$

c)
$$T\,dS = dE + P\,dV - \mu\,dQ$$

$$-S\,dT = d(E - TS) - P\,dV - \mu\,dQ$$

$$-S\,dT = d(E - TS - PV) + V\,dP - \mu\,dQ$$

$$-S\,dT = d(E - TS - PV - \mu Q) + V\,dP + Q\,d\mu$$

$$-S\,dT = d\underbrace{(E - TS - PV - \mu Q)}_{= \emptyset}$$

$$\frac{\partial P}{\partial T}\Big|_\mu = S/V \quad \checkmark$$

6. Beginning with:

$$dE = TdS - PdV + \mu dQ,$$

derive the Maxwell relation,

$$\left.\frac{\partial V}{\partial \mu}\right|_{S,P} = -\left.\frac{\partial N}{\partial P}\right|_{S,\mu}.$$

$$d(E + PV - \mu N) = TdS + VdP - Nd\mu$$

$$\left.\frac{\partial N}{\partial P}\right|_{S,\mu} = \left.\frac{\partial V}{\partial \mu}\right|_{P,S}$$

✓

7. Beginning with the definition,

$$C_P = T\frac{\partial S}{\partial T}\bigg|_{N,P},$$

Show that

$$\frac{\partial C_p}{\partial P}\bigg|_{T,N} = -T\,\frac{\partial^2 V}{\partial T^2}\bigg|_{P,N}$$

Hint: Find a quantity $\boldsymbol{Y}$ for which both sides of the equation become $\partial^3 Y/\partial^2 T \partial P$.

8. Beginning with

show $T\dfrac{\partial^2 S}{\partial T\,\partial P}\bigg|_N = -T\dfrac{\partial^2 V}{\partial T^2}\bigg|_N$

$$dE = TdS - PdV + \mu dN$$

$$d\underbrace{(E - TS + PV)}_{Y\,=\,G} = -\underset{\uparrow\ \frac{\partial Y}{\partial T}}{S}\,dT + \underset{\uparrow\ \frac{\partial Y}{\partial P}}{V}\,dP + \mu dN$$

$$\frac{\partial^3 Y}{\partial T^2 \partial P} = \frac{\partial^3 Y}{\partial T \partial P \partial T}$$

$$= \frac{\partial^2 V}{\partial T^2} = -\frac{\partial^2 S}{\partial T \partial P}$$

✓

8. Beginning with

$$TdS = dE + PdV - \mu dQ, \quad \text{and } G \equiv E + PV - TS,$$

(a) Show that

$$S = -\left.\frac{\partial G}{\partial T}\right|_{N,P}, \quad V = \left.\frac{\partial G}{\partial P}\right|_{N,T}.$$

(b) Beginning with

$$\delta S(P,N,T) = \frac{\partial S}{\partial P}\delta P + \frac{\partial S}{\partial N}\delta N + \frac{\partial S}{\partial T}\delta T,$$

Show that the specific heats,

$$C_P \equiv T\left.\frac{\partial S}{\partial T}\right|_{N,P}, \quad C_V \equiv T\left.\frac{\partial S}{\partial T}\right|_{N,V},$$

satisfy the relation:

$$C_P = C_V - T\left(\left.\frac{\partial V}{\partial T}\right|_{P,N}\right)^2\left(\left.\frac{\partial V}{\partial P}\right|_{T,N}\right)^{-1}$$

Note that the compressibility, $\equiv -\partial V/\partial P$, is positive (unless the system is unstable), therefore $C_P > C_V$.

a) $dE = TdS - PdV + \mu dQ$

$dG = -SdT + VdP + \mu dQ$

$$S = -\left.\frac{\partial G}{\partial T}\right|_{P,Q} \qquad V = \left.\frac{\partial G}{\partial P}\right|_{TQ}$$

b) $$\delta S = \left.\frac{\partial S}{\partial P}\right|_{T,N}\delta P + \left.\frac{\partial S}{\partial T}\right|_{P,N}\delta T + \cancel{\frac{\partial S}{\partial N}\delta N}$$ keep fixed

$$\delta V = \left.\frac{\partial V}{\partial P}\right|_{T,N}\delta P + \left.\frac{\partial V}{\partial T}\right|_{P,N}\delta T + \cancel{\frac{\partial V}{\partial N}\delta N} = \emptyset$$

$$\delta P = -\left.\frac{\partial V}{\partial T}\right|_{P,N}\cdot\delta T \Big/ \left.(\partial V/\partial P)\right|_{T,N}$$

$$\delta S\Big|_V = -\left[\left.\frac{\partial S}{\partial P}\right|_{T,N}\cdot\left.\frac{\partial V}{\partial T}\right|_{P,N}\Big/\left.\frac{\partial V}{\partial P}\right|_{T,N}\right]\delta T$$

$$+\left.\frac{\partial S}{\partial T}\right|_{P,N}\delta T$$

$$C_V = C_P - T\frac{\partial S/\partial P|_T \; \partial V/\partial T|_P}{\partial V/\partial P|_T}$$

Next, show

$$\left.\frac{\partial S}{\partial P}\right|_{T,N} = -\left.\frac{\partial V}{\partial T}\right|_{P,N}$$

$$dE = T\,dS + \mu\,dN - P\,dV$$

$$d(E - TS + PV) = \mu\,dN - S\,dT + V\,dP$$

$$\left.\frac{\partial S}{\partial P}\right|_{N,T} = -\left.\frac{\partial V}{\partial T}\right|_{P,N} \quad ✓$$

from previous page,

$$C_V = C_P + T\,\frac{\left(\partial V/\partial T|_{P,N}\right)^2}{\partial V/\partial P|_{T,N}}$$

$$C_P = C_V - T\,\frac{\left(\partial V/\partial T|_{T,N}\right)^2}{\partial V/\partial P|_{T,N}}$$

9. From Sec. 1.11, it was shown how to derive fluctuations in the grand canonical ensemble. Thus, it is straightforward to find expressions for the following fluctuations, $\phi_{EE} \equiv \langle \delta E \delta E \rangle / V$, $\phi_{QQ} \equiv \langle \delta Q \delta Q \rangle / V$ and $\phi_{QE} \equiv \langle \delta E \delta Q \rangle / V$. In terms of the 3 fluctuations above, calculated in the grand canonical ensemble, and in terms of the volume and the temperature T, express the specific heat at constant volume and charge,

$$C_V = \left. \frac{dE}{dT} \right|_{Q,V}.$$

Note that the fluctuation observables, ϕ_{ij}, are intrinsic quantities, assuming that the correlations in energy density occur within a finite range and that the overall volume is much larger than that range.

$$\phi_{EE} = \frac{1}{V} \frac{\partial^2 \ln Z}{\partial \beta^2}, \quad \varphi_{QE} = \frac{1}{V} \frac{\partial^2 \ln Z}{\partial \beta \partial \alpha}, \quad \varphi_{QQ} = \frac{\partial^2 \ln Z}{\partial \alpha^2} \frac{1}{V}$$

$$\ln Z = \beta P V$$

$$\varphi_{EE} = \frac{\partial^2 (\beta P)}{\partial \beta^2}; \quad \varphi_{QE} = \frac{\partial^2 (\beta P)}{\partial \beta \partial \alpha}, \quad \varphi_{QQ} = \frac{\partial^2 (\beta P)}{\partial \alpha^2}$$

$$\varphi_{EE} = -\frac{\partial E}{\partial \beta} \frac{1}{V}, \quad \varphi_{QE} = -\frac{\partial E}{\partial \alpha} \frac{1}{V} = -\frac{\partial Q}{\partial E}, \quad \varphi_{QQ} = \frac{\partial Q}{\partial \alpha} \frac{1}{V}$$

$$\delta(Q/V) = \frac{\partial (\beta P)}{\partial \alpha} \delta \alpha + \frac{\partial (\beta P)}{\partial \beta} \delta \beta = \emptyset$$

$$\delta E \Big|_{V,Q} = \frac{\partial E}{\partial \alpha} \delta \alpha + \frac{\partial E}{\partial \beta} \delta \beta$$

$$\delta Q \Big|_{V} = \frac{\partial Q}{\partial \alpha} \delta \alpha + \frac{\partial Q}{\partial \beta} \delta \beta = \emptyset$$

$$\delta E \Big|_{V,Q} = \left[-\frac{\partial E}{\partial \alpha} \frac{\partial Q / \partial \beta}{\partial Q / \partial \alpha} + \frac{\partial E}{\partial \beta} \right] \delta \beta$$

$$\delta E \Big|_{V,Q} = \left[V \varphi_{QE}^2 / \varphi_{QQ} - V \varphi_{EE} \right] \delta \beta, \quad \delta \beta = -\frac{\delta T}{T^2}$$

$$C_V = \frac{V}{T^2} \left\{ \varphi_{EE} - \frac{\varphi_{QE}^2}{\varphi_{QE}} \right\}$$

2 Chapter 2

1. Consider classical non-relativistic particles acting through a spherically symmetric potential,

$$V(r) = V_0 \exp(r/\lambda).$$

Using the equipartion and virial theorems, show that

$$\left\langle \frac{r}{\lambda} V(r) \right\rangle = 3T.$$

$$\left\langle x \left.\frac{\partial V}{\partial x}\right|_{y,z} \right\rangle = \left\langle y \left.\frac{\partial V}{\partial y}\right|_{x,z} \right\rangle = \left\langle z \left.\frac{\partial V}{\partial z}\right|_{x,y} \right\rangle$$

$$= \left\langle p_x \frac{\partial KE}{\partial p_x} \right\rangle \cdots = T \leftarrow \text{equipart.th}$$

virial

$$\left\langle r \left.\frac{\partial V}{\partial r}\right|_{\theta,\varphi} \right\rangle = \left\langle \vec{r} \cdot \vec{\nabla} V \right\rangle = \left\langle x \frac{\partial V}{\partial x} + y \frac{\partial V}{\partial y} + z \frac{\partial V}{\partial z} \right\rangle$$

$$= 3T$$

2. Consider a relativistic ($\epsilon = \sqrt{m^2 + p^2}$) particle moving in one dimension,

(a) Using the generalized equipartion theorem, show that

$$\left\langle \frac{p^2}{\epsilon} \right\rangle = T.$$

(b) Show the same result by explicitly performing the integrals in

$$\left\langle \frac{p^2}{\epsilon} \right\rangle = \frac{\int dp\, (p^2/\epsilon) e^{-\epsilon/T}}{\int dp\, e^{-\epsilon/T}}.$$

HINT: Integrate the numerator by parts.

a) $\left\langle p \frac{\partial H}{\partial p} \right\rangle = T = \left\langle p \frac{\partial}{\partial p} \sqrt{p^2+m^2} \right\rangle$

$= \left\langle p^2/\varepsilon \right\rangle$

b) $\int dp\, (p^2/\varepsilon)\, e^{-\varepsilon/T}$

$= -T \int dp \cdot p \,\frac{d}{dp}\left(e^{-\varepsilon/T}\right)$

$= T \int dp\, e^{-\varepsilon/T}$

$\left\langle \frac{p^2}{\varepsilon} \right\rangle = T$ ✓

3. Beginning with the expression for the pressure for a non-interacting gas of bosons,

$$\frac{PV}{T} = \ln Z_{GC} = \sum_p \ln\left(1 + e^{-\beta(\epsilon_p-\mu)} + e^{-2\beta(\epsilon_p-\mu)} + \cdots\right), \sum_p \rightarrow (2s+1)\frac{V}{(2\pi\hbar)^3}\int d^3p,$$

show that

$$P = \frac{(2s+1)}{(2\pi\hbar)^3}\int d^3p \frac{p^2}{3\epsilon} f(\mathrm{p}), \text{ where } f = \frac{e^{-\beta(\epsilon-\mu)}}{1-e^{-\beta(\epsilon-\mu)}}.$$

Here, the energy is relativistic, $\epsilon = \sqrt{p^2+m^2}$.

$$\frac{PV}{T} = \frac{(2s+1)V}{(2\pi\hbar)^3}\int d^3p \ln\left[\frac{1}{1-e^{-\beta(\varepsilon-\mu)}}\right]$$

$$= -4\pi\frac{(2s+1)V}{(2\pi\hbar)^3}\int p^2 dp \ln\left(1-e^{-\beta(\varepsilon-\mu)}\right)$$

$$= 4\pi\frac{(2s+1)V}{(2\pi\hbar)^3}\int \frac{p^3}{3} dp \frac{d}{dp}\ln(1 - \cdots)$$

$$= \frac{4\pi(2s+1)V}{(2\pi\hbar)^3}\int \frac{p^4}{3\varepsilon T} \frac{e^{-\beta(\varepsilon-\mu)}}{1-e^{-\beta(\varepsilon-\mu)}}$$

$$P = \frac{(2s+1)}{(2\pi\hbar)^3}\int d^3p \frac{p^2}{3\varepsilon}\frac{e^{-\beta(\varepsilon-\mu)}}{1-e^{-\beta(\varepsilon-\mu)}}$$

4. Derive the corresponding expression for Fermions in the previous problem.

$$\frac{PV}{T} = 4\pi \frac{(2s+1)V}{(2\pi\hbar)^3} \int p^2 dp \ln\left(1 + e^{-\beta(\varepsilon - \mu)}\right)$$

$$= 4\pi \frac{(2s+1)V}{(2\pi\hbar)^3} \int \frac{p^3}{3} dp \frac{d}{dp} \ln(1 + \cdots)$$

$$= \frac{4\pi (2s+1)V}{(2\pi\hbar)^3} \int \frac{p^4}{3\varepsilon T} \frac{e^{-\beta(\varepsilon-\mu)}}{1+e^{-\beta(\varepsilon-\mu)}}$$

$$P = \frac{(2s+1)}{(2\pi\hbar)^3} \int d^3p \frac{p^2}{3\varepsilon} \frac{e^{-\beta(\varepsilon-\mu)}}{1+e^{-\beta(\varepsilon-\mu)}}$$

5. Derive the corresponding expression for Bosons/Fermions in two dimensions in the previous problem. Note that in two dimension, $\boldsymbol{P}$ describes the work done per expanding by a unit area, $\boldsymbol{dW} = \boldsymbol{PdA}$.

$$\frac{P A}{T} = \frac{(2s+1)A}{(2\pi\hbar)^2} \int 2\pi p\, dp \mp \ln\left(1 \mp e^{-\beta\varepsilon - \mu}\right)$$

$$= \frac{(2s+1)A}{(2\pi\hbar)^2} (\pm) \int dp \frac{p^2}{2} \frac{d}{dp} \ln\left(1 \mp e^{-\beta(\varepsilon-\mu)}\right) \cdot 2\pi$$

$$P A = \frac{(2s+1)A}{(2\pi\hbar)^2} \int d^2p \left(\frac{p^2}{2\varepsilon}\right) \frac{e^{-\beta(\varepsilon-\mu)}}{1 - e^{-\beta(\varepsilon-\mu)}}$$

$$P = \frac{(2s+1)}{(2\pi\hbar)^2} \int d^3p \left(\frac{p^2}{2\varepsilon}\right) f(\vec{p})$$

6. For the two-dimensional problem above, show that $\boldsymbol{P}$ gives the rate at which momentum is transferred per unit length of the boundary of a 2-d confining box. Use simple kinematic considerations.

collision rate $= \dfrac{N \cdot |v_x|}{2 L_y}$ $\quad$ N is ~~#~~ of molecules

momentum transfer rate $=$

collision rate $\cdot |2 p_x|$

momentum transfer rate on one face of box $= N \cdot \dfrac{\langle p_x^2 \rangle}{m L_y} = \dfrac{N \langle p^2 \rangle}{2 m L_y}$

"Pressure" $=$ momentum transfer rate $/ L_x$

$$= \frac{N}{A} \left\langle \frac{p^2}{2m} \right\rangle$$

$$N \left\langle \frac{p^2}{2m} \right\rangle = (2s+1) \int \frac{d^2 p \, A}{(2\pi\hbar)^2} f(p) \frac{p^2}{2m}$$

$$\text{"Pressure"} = (2s+1) \int \frac{d^2 p}{(2\pi\hbar)^2} f(\vec{p}) \frac{p^2}{2m}$$

✓

7. Consider a massless three-dimensional gas of bosons with spin degeneracy N_s. Assuming zero chemical potential, find the coefficients A and B for the expressions for the pressure and energy density,

$$P = AN_sT^4, \quad \left(\frac{E}{V}\right) = BN_sT^4$$

$$P = N_s \frac{1}{(2\pi\hbar)^3} \int 4\pi p^2 dp \left(\frac{e^{-\beta p}}{1-e^{-\beta p}}\right) \frac{p^2}{3p}$$

$$= \frac{N_s}{6\pi^2\hbar^3} \int p^3 dp \left\{ e^{-\beta p} + e^{-2\beta p} + e^{-3\beta p} + \cdots \right\}$$

$$= \frac{N_s T^4}{6\pi^2\hbar^3} \cdot 3! \left\{ 1 + \left(\frac{1}{2}\right)^4 + \left(\frac{1}{3}\right)^4 + \cdots \right\}$$

$$= \frac{N_s T^4}{\pi^2\hbar^3} \zeta(4)$$

$$\uparrow = \pi^4/90$$

$$P = \frac{\pi^2 T^4}{90\hbar^3} N_s, \quad A = \frac{\pi^2}{90\hbar^3}, \quad B = 3A$$

8. Show that if the previous problem is repeated for Fermions that:

$$A_{Fermions} = \frac{7}{8} A_{Bosons}, \quad B_{Fermions} = \frac{7}{8} B_{Bosons}.$$

For fermions,

$$P = \frac{N_s T^4}{\pi^2 \hbar^3} \left\{ 1 - \left(\frac{1}{2}\right)^4 + \left(\frac{1}{3}\right)^4 - \left(\frac{1}{4}\right)^4 \cdots \right\}$$

$$= P_{BOSONS} - \frac{2 N_s T^4}{\pi^2 \hbar^3} \left\{ \left(\frac{1}{2}\right)^4 + \left(\frac{1}{4}\right)^4 + \left(\frac{1}{6}\right)^4 \cdots \right\}$$

$$= P_{BOSONS} - P_{BOSONS} \cdot \frac{1}{8}$$

$$= \frac{7}{8} P_{BOSONS}$$

$$A_{FERMIONS} = \frac{7}{8} P_{BOSONS}$$

$$B_{FERMIONS} = \frac{7}{8} B_{BOSONS}$$

9. Consider a three-dimensional solid at low temperature where both the longitudinal and transverse sound speeds are given by $c_s = 3000$ m/s. Calculate the ratio of specific heats,

$$\frac{C_V(\text{due to phonons})}{C_V(\text{due to photons})},$$

where the photon calculation assumes the photons move in a vacuum of the same volume. Note that for sound waves the energy is $\epsilon = \hbar\omega = c_s\hbar k = c_s p$. For phonons, there are three polarizations (two transverse and one longitudinal), and since the temperature is low, one can ignore the Debye cutoff which excludes high momentum nodes, as their wavelengths are below the spacing of the crystal.
Data: $c = 3.0 \times 10^8$ m/s, $\hbar = 1.05457266 \times 10^{-34}$ Js.

$$E/V = \frac{\pi^2}{30(\hbar c)^3} \cdot 2\, T^4 \qquad \text{photons}$$

$$= \frac{\pi^2}{30(\hbar c_s)^3} \cdot 3\, T^4 \qquad \text{phonons}$$

$$C_V = \frac{4}{T} E/V$$

$$\frac{C_V(\text{phonons})}{C_V(\text{photons})} = \frac{3}{2}\left(\frac{c}{c_s}\right)^3$$

$$= 1.5 \cdot 10^{15}$$

10. For a one-dimensional non-relativistic gas of spin-1/2 Fermions of mass m, find the change of the chemical potential $\delta\mu(T,\rho)$ necessary to maintain a constant density per unity length, ρ, while the temperature is raised from zero to T. Give answer to order T^2.

$$\delta\rho = \frac{1}{A}\frac{dD}{d\varepsilon}\frac{\pi^2}{6}T^2 + \frac{D}{A}\delta\mu = 0$$

$$\frac{1}{A}D = 2\cdot\frac{1}{(2\pi\hbar)}\cdot 2\frac{m}{p} = \frac{2}{\pi\hbar}\frac{1}{\sqrt{2mE}}$$

$$\delta\mu = -\frac{dD}{d\varepsilon}\cdot\frac{\pi^2}{6}\frac{T^2}{D}$$

$$\frac{dD}{d\varepsilon} = -\frac{1}{2}\frac{D}{\varepsilon_F}$$

$$\boxed{\delta\mu = \frac{\pi^2}{12}\frac{T^2}{\varepsilon_F}}$$

$$\rho = \frac{2}{\pi\hbar}p_f = \frac{2}{\pi\hbar}\sqrt{2m\varepsilon_F}$$

$$\varepsilon_F = \left(\frac{\pi\hbar\rho}{2}\right)^2\frac{1}{2m}$$

$$\delta\mu = \frac{\pi^2}{12}\frac{T^2}{\varepsilon_F}, \quad \varepsilon_F = \frac{\pi^2\hbar^2\rho^2}{8m}$$

11. For a two-dimensional gas of spin-1/2 non-relativistic Fermions of mass m at low temperature, find both the quantities below:

$$\left.\frac{d(E/A)}{dT}\right|_{\mu,V}, \quad \left.\frac{d(E/A)}{dT}\right|_{N,V}$$

Give both answers to the lowest non-zero order in T, providing the constants of proportionality in terms of the chemical potential at zero temperature and m.

$$\left.\frac{1}{A} E^{*}\right|_{N} = \frac{\pi^2}{6}\left(D/A\right)T^2$$

$$\left.\frac{1}{A} E^{*}\right|_{\mu} = \frac{\pi^2}{6}\left(\frac{D}{A}\right)T^2 + \mu \Delta N\Big|_{\mu}$$

$$\Delta N\Big|_{\mu} = \frac{\pi^2}{6}\left(\frac{D'}{A}\right)T^2 = 0 \quad \text{because } D' = 0 \,!$$

$$, 0$$

$$\left.\frac{d(E/A)}{dT}\right|_{\mu,V} = \left.\frac{d(E/A)}{dT}\right|_{N,V} \quad \text{in this case}$$

$$= \frac{\pi^2}{3}\left(\frac{D}{A}\right)\cdot T$$

$$D/A = \frac{2}{(2\pi\hbar)^2}\cdot 2\pi \cancel{p_f}\, \frac{m}{\cancel{p_f}}$$

$$= \frac{m}{\pi \hbar^2}$$

$$\left.\frac{d(E/A)}{dT}\right|_{\mu,V} = \left.\frac{d(E/A)}{dT}\right|_{N,V} = \frac{\pi m}{3\hbar^2} T$$

12. The neutron star, PSR J1748-2446ad, discovered in 2004, spins at 716 times a second. Spin half particles have a spin angular momentum of $S_z = \pm\hbar/2$. If the neutron star has a temperature of 10^5 K, what is the polarization due to the spinning? $P = (n_\uparrow - n_\downarrow)/(n_\uparrow + n_\downarrow)$. Note that this neglects the polarization due to the magnetic field.

$$P = \frac{e^{+(\beta\hbar\omega/2)} - e^{-(\beta\hbar\omega/2)}}{e^{\beta\hbar\omega/2} + e^{-\beta\hbar\omega/2}}$$

$$= \tanh(\beta\hbar\omega/2)$$

$$T = 10^5\ \mathrm{K} = 1.38 \cdot 10^{-18}\ \mathrm{J}$$

$$\hbar = 1.055 \cdot 10^{-34}\ \mathrm{J \cdot s}$$

$$\beta\hbar\omega/2 = \frac{1}{2}\,\frac{1.055 \cdot 10^{-34}}{1.38 \cdot 10^{-18}} \cdot 2\pi \cdot 716$$

$$= 1.72 \cdot 10^{-13}$$

$$P = 1.72 \cdot 10^{-13}$$

3 Chapter 3

3.6 Problems

1. Consider a low density three-dimensional gas of non-relativistic spin-zero bosons of mass m at temperature $T = 1/\beta$ and chemical potential μ.

   (a) Find ρ_0 as defined in Eq. (3.1) in terms of m and T.

   (b) Expand the density ρ to second order in $e^{\beta\mu}$, i.e., to $e^{2\beta\mu}$. Express your answers for this part and the next two parts in terms of ρ_0.

   (c) Expand ρ^2 to second order in $e^{\beta\mu}$.

   (d) Expand $\delta P \equiv P - \rho T$ to second order in $e^{\beta\mu}$. (Hint: it is easier if you use the expression for P expanded in $f_0 = e^{\beta\mu-\beta\epsilon}$, i.e., $\ln(1 + f_0 + f_0^2 + f_0^3 \cdots) = -\ln(1 - f_0)$, then expand the logarithm in powers of f_0)

   (e) Determine the second virial coefficient defined by Eq. (3.1).

a) $$\rho_0 = \frac{1}{(2\pi\hbar)^3}\left[\int dp\, e^{-p^2/2mT}\right]^3$$

$$= \frac{(mT)^{3/2}}{(2\pi\hbar^2)^{3/2}}$$

b) $$\rho = \frac{1}{(2\pi\hbar)^3}\int d^3p\left\{e^{-p^2/2mT+\beta\mu} + e^{-2p^2/2mT+2\beta\mu} + e^{-3\ldots} \cdots\right\}$$

$$= e^{\beta\mu}\cdot\rho_0 + e^{2\beta\mu}\rho_0\cdot\left(\frac{1}{2}\right)^{3/2}$$

because $T \to T/2$ in integral

c) $$\rho^2 = e^{2\beta\mu}\rho_0^2 + \cdots$$

d) $$P = \frac{T}{(2\pi\hbar)^2}\int d^3p \ln\left\{1 + e^{-\beta(\varepsilon-\mu)} + e^{-2\beta(\varepsilon-\mu)} + \cdots\right\}$$

$$= \frac{T}{(2\pi\hbar)^2}\int d^3p\left\{e^{-\beta(\varepsilon-\mu)} + e^{-2\beta(\varepsilon-\mu)} + e^{-3\beta(\varepsilon-\mu)} - \frac{1}{2}e^{-2\beta\varepsilon-\mu} + \cdots\right\}$$

$$\simeq \frac{T}{(2\pi\hbar)^2}\int d^3p\left\{e^{-\beta(\varepsilon-\mu)} + \frac{1}{2}e^{-2\beta(\varepsilon-\mu)} + \cdots\right\}$$

$$P/T = \rho - \frac{1}{2}e^{2\beta\mu}\rho_0\left(\frac{1}{2}\right)^{3/2}$$

$$= \rho - \left(\frac{1}{2}\right)^{5/2}e^{2\beta\mu}\rho_0^2/\rho_0 + \cdots$$

$$\simeq \rho - \left(\frac{1}{2}\right)^{5/2}\rho^2/\rho_0, \quad A_2 = -\left(\frac{1}{2}\right)^{5/2}$$

2. Consider the Van der Waals equation of state in scaled variables,

$$p = \frac{t}{v-1} - \frac{1}{v^2},$$

where $p = P/a\rho_s^2$, $v = V/V_s$, $t = T/a\rho_s$.

(a) Derive the Maxwell relation,

$$\left.\frac{\partial(P/T)}{\partial\beta}\right|_{N,V} = -\left.\frac{\partial E}{\partial V}\right|_{N,T}.$$

(b) Find the scaled energy per particle $e \equiv E/(a\rho_s N)$ as a function of v and t using the Maxwell relation above. Begin with the fact that $e = (3/2)t$ as $v \to \infty$.

(c) Show that the change of entropy/particle $s = S/N$ between two values of v at a fixed temperature t is:

$$s_b - s_a = \ln[(v_b - 1)/(v_a - 1)].$$

(d) Using the fact that $ts = e + pv - \mu$, show that

$$\mu_b - \mu_a = -\frac{2}{v_b} + \frac{2}{v_a} + t\left[\frac{v_b}{v_b - 1} - \frac{v_a}{v_a - 1}\right] - t\ln\left(\frac{v_b - 1}{v_a - 1}\right).$$

(e) Show that as $t \to 0$, p_b will equal p_a if $v_b \to \infty$ and $v_a = 1 + t$. Then, show that in the same limit, μ_a will equal μ_b if $v_b = te^{1/t}$.

47

a) $dS = \beta dE + (\beta P) dV - \beta\mu dN$

$d(S - \beta E) = -E d\beta + \beta P dV - \beta\mu dN$

$$\left.\frac{\partial \beta P}{\partial\beta}\right|_{N,V} = -\left.\frac{\partial E}{\partial V}\right|_{N,T}$$

b) $$e = \frac{3t}{2} - \int_\infty^v dv \left.\frac{\partial(\beta P)}{\partial\beta}\right|_v = \frac{3t}{2} - \int_v^\infty dv\, \frac{1}{v^2}$$

$$= \frac{3t}{2} - \frac{1}{v}$$

c) $ds = \beta de + \beta p dv$ (N is fixed)

$= \left(\beta \frac{1}{v^2} + \beta p\right) dv$

$= \left(\frac{1}{v-1}\right) dv$

$$s_b - s_a \int_{v_a}^{v_b} dv\, \frac{1}{v-1} = \ln\left(\frac{v_b - 1}{v_a - 1}\right)$$

d) $\mu_b - \mu_a = -t(s_b - s_a) + e_b - e_a + p_b v_b - p_a v_a$

$$= -t\ln\left(\frac{v_b - 1}{v_a - 1}\right) - \frac{1}{v_b} + \frac{1}{v_a} + t\left(\frac{v_b}{v_b - 1} - \frac{v_a}{v_a - 1}\right) - \frac{1}{v_b} + \frac{1}{v_a}$$

$$= -t\ln\left(\frac{v_b - 1}{v_a - 1}\right) - \frac{2}{v_b} + \frac{2}{v_a} + t\left(\frac{v_b}{v_b - 1} - \frac{v_a}{v_a - 1}\right)$$

e) $p_b - p_a = \frac{t}{v_b - 1} - \frac{1}{v_b^2} - \frac{t}{v_a - 1} + \frac{1}{v_a^2}$

If $v_b \to \infty$, $t \to 0$,

$p_b - p_a = \frac{t}{v_a - 1} + \frac{1}{v_a^2}$, $= 0$ if $t = \frac{v_a - 1}{v_a^2}$

$\underline{v_a = 1 + t}$ as $T \to \infty$

$\mu_b - \mu_a = \frac{2}{1+t} + t\left[1 - \frac{1}{t}\right] - t \ln\left(\frac{v_b}{t}\right)$

$\simeq 1 - t \ln(v_b/t)$

$= 0$ if $\ln(v_b/t) = 1/t$

$v_b = t e^{1/t}$ as $t \to \infty$

(f) Find the latent heat $L = t(s_b - s_a)$ for the small t limit. How does it compare with the minimum of e at $t = 0$?

(g) At $t = 0$, the system will have $p = 0$ in order to minimize the energy. Using the Clausius-Clapeyron equation, find dp/dt along the coexistence line at $t = 0$.

f) $L = t \ln\left(\frac{v_b - 1}{v_a - 1}\right) \simeq t \ln \cancel{v_b}/\cancel{t} = 1$

$e(t=0) = -\frac{1}{v} = -1$, $L = -e$

g) $\frac{dp}{dt} = \frac{L}{v_b - v_a} = \frac{1}{t e^{1/t}} = \emptyset$

3. Using Eq. (3.23), calculate the second-order virial coefficient for a gas of distinguishable non-relativistic particles of mass m at temperature T that interact through a hard core potential,

$$V(r) = \begin{cases} \infty\,, & r < a \\ 0\,, & r > a \end{cases}$$

Consider only the s-wave contribution (valid at low T).

$$\psi(r) \sim \sin(kr + \delta)$$

$$ka + \delta = 0\,, \quad \delta = -ka$$

the definition of the virial expansion coefficients,

$$A_2 = -2^{3/2} \sum_\ell \int d\epsilon\, \frac{(2\ell+1)}{\pi} \frac{d\delta_\ell}{d\epsilon} e^{-\epsilon/T}.$$

$$\varepsilon = \frac{k^2}{2m}\,,$$

$$A_2 = -(2^{3/2}) \frac{1}{\pi} \int dk\, \frac{d\delta}{dk}\, e^{-k^2/2mT}$$

$$= -(2^{3/2}) \frac{a}{\pi} \int dk\, e^{-k^2/2mT}$$

$$= -4a\left(\frac{mT}{\pi}\right)^{1/2}$$

4 Chapter 4

1. A molecule of mass m has two internal states, a spin-zero ground state and a spin-1 excited state which is at energy X above the ground state. Initially, a gas of such molecules is at temperature T_i before expanding and cooling isentropically to a temperature T_f. Neglect quantum degeneracy of the momentum states for the following questions.

   (a) What is the initial energy per particle? Give answer in terms of m, T_i, X and the initial density ρ_i.

   (b) Derive an expression for the initial entropy per particle in terms of the same variables.

   (c) After insentropically cooling to T_f, find the density ρ_f. Give answer in terms of ρ_i, T_i, T_f and X.

a) $$E = \frac{3}{2}T + \frac{3Xe^{-\beta X}}{1+3e^{-\beta X}}$$

b) $$S/N = \ln Z + \beta E$$

$$= \frac{5}{2} + \ln\left\{\rho\frac{(mT)^{3/2}}{(2\pi\hbar^2)^{3/2}}\right\} + \sigma_{\text{internal}}$$

$$\sigma_{\text{internal}} = \frac{3\beta X e^{-\beta X}}{1+3e^{-\beta X}} + \ln\left(1+3e^{-\beta X}\right)$$

$$\ln \rho_i T_i^{3/2} + \frac{3\beta_i X e^{-\beta_i X}}{1+3e^{-\beta_i X}} + \ln\left(1+3e^{-\beta_i X}\right)$$

$$= \ln \rho_f T_f^{3/2} + \frac{3\beta_f X e^{-\beta_f X}}{1+3e^{-\beta_f X}} + \ln\left(1+3e^{-\beta_f X}\right)$$

$$\rho_f = T_f^{-3/2}\exp\left\{\ln\rho_i T_i^{3/2} + \frac{3\beta_i X e^{-\beta_i X}}{1+3e^{-\beta_i X}} + \ln\left(1+3e^{-\beta_i X}\right) - \frac{3\beta_f X e^{-\beta_f X}}{1+3e^{-\beta_f X}} + \ln\left(1+3e^{-\beta_f X}\right)\right.$$

$$\rho_f = \rho_i\left(\frac{T_i}{T_f}\right)^{3/2}\exp\left\{\sigma_{int}(T_i) - \sigma_{int}(T_f)\right\}$$

$$\sigma_{int} = \frac{3\beta X e^{-\beta X}}{1+3e^{-\beta X}} + \ln\left(1+3e^{-\beta X}\right)$$

2. Repeat problem #1 above, but assume the molecule has the excitation spectrum of a 3-dimensional harmonic oscillator, where the energy levels are separated by amounts $\hbar\omega = \boldsymbol{X}$, with $\boldsymbol{X} << \boldsymbol{T}$.

$$\sigma_{int} = \ln z_{int} + \beta \varepsilon_{int}.$$

$$= -\ln (1 - e^{-\beta X})^3 + \beta X \frac{e^{-\beta X}}{1 - e^{-\beta X}}$$

$$= -3 \ln (1 - e^{-\beta X}) + \beta X \frac{e^{-\beta X}}{1 - e^{-\beta X}}$$

$$\rho_f = \rho_i \left(\frac{T_i}{T_f}\right)^{3/2} \exp\left\{\sigma_{int}(T_i) - \sigma_{int}(T_f)\right\}$$

3. A large number of N diatomic molecules of mass m are confined to a region by a harmonic-oscillator potential,

$$V(\vec{r}) = \frac{1}{2}kr^2.$$

The system is at a sufficient temperature T so that the gas can be considered dilute and the energy levels are practically continuous. The temperature is in the range where rotational modes are routinely excited, but vibrational modes can be neglected.

(a) (10 pts) What is the energy per particle? Give your answer in terms of m, T, and k.

(b) (10 pts) Derive an expression for the entropy per particle in terms of the same variables. Begin with the expression,

$$S = \ln Z + \beta E,$$

where

$$Z = \frac{z^N}{N!},$$

and z is the partition function of a single molecule.

(c) (10 pts) If the spring constant is adiabatically changed from k_i to k_f, and if the initial temperature is T_i, find T_f.

a

$$E/N = 4T$$

$$= \left(\frac{3}{2}T + T + \frac{3}{2}T\right)$$

b

$$z = z_{rot} \cdot z_{h.o}$$

$$z_{rot} = 2IT/\hbar^2$$

$$z_{h.o.} = (z_x)^3, \quad z_x = \frac{1}{1-e^{-\beta\hbar\omega}}, \quad \omega = \sqrt{k/m}$$

$$\frac{S}{N} = \frac{\ln Z}{N} + \beta E/N, \quad Z = \frac{z^N}{N!}, \ \ln N! = N\ln N - N$$

$$= \ln z - \frac{\ln N!}{N} + 4$$

$$= \ln(z/N) + 5$$

$$\ln z = \ln z_{rot} + 3\ln z_x = \ln\frac{2IT}{\hbar^2} - 3\ln\left(1-e^{-\beta\hbar\omega}\right)$$

$\beta\hbar\omega << 1$, so

$$\ln z = \ln\left(\frac{2IT}{\hbar^2}\right) - 3\ln(\beta\hbar\omega) = \ln\frac{2IT^4}{\hbar^5\omega^3}$$

$$\frac{S}{N} = \ln\left(\frac{2IT^4}{\hbar^5\omega^3 N}\right) + 5, \quad \omega = \sqrt{k/m}$$

$$\frac{S}{N} = \ln\left(\frac{2IT^4 m^{3/2}}{\hbar^5 k^{3/2} N}\right)$$

c

$$\frac{T_i^5}{k_i^{3/2}} = \frac{T_f^5}{k_f^{3/2}}, \quad T_f = T_i\left(\frac{k_f}{k_i}\right)^{3/10}$$

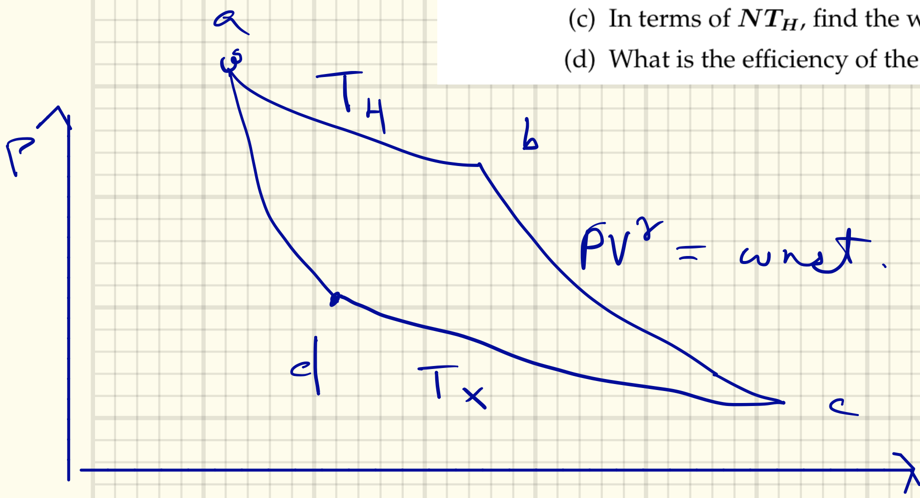
4. Consider an ideal gas with $C_p/C_v = \gamma$ going through the Carnot cycle illustrated in Fig. 4.1. The initial volume for N molecules at temperature T_H expands from V_a to $V_b = 2V_a$ and then to $V_c = 2V_b$.

(a) In terms of NT_H, find the work done while expanding from $a - b$.

(b) Again, in terms of NT_H, how much heat was added to the gas while expanding from $a - b$.

(c) In terms of NT_H, find the work done while expanding from $b - c$.

(d) What is the efficiency of the cycle ($abcd$)?



a) $W_{ab} = \int_{V_a}^{V_b} P\,dV = N T_H \int_{V_a}^{V_b} \frac{dV}{V} = N T_H \ln 2$

b) $E = C_v T$,

$E_b = E_a$, so $Q = W_{ab} = N T_H \ln 2$

c) $W_{bc} = \int_{V_b}^{V_c} P\,dV = P_b \int \left(\frac{V_b}{V}\right)^\gamma dV$

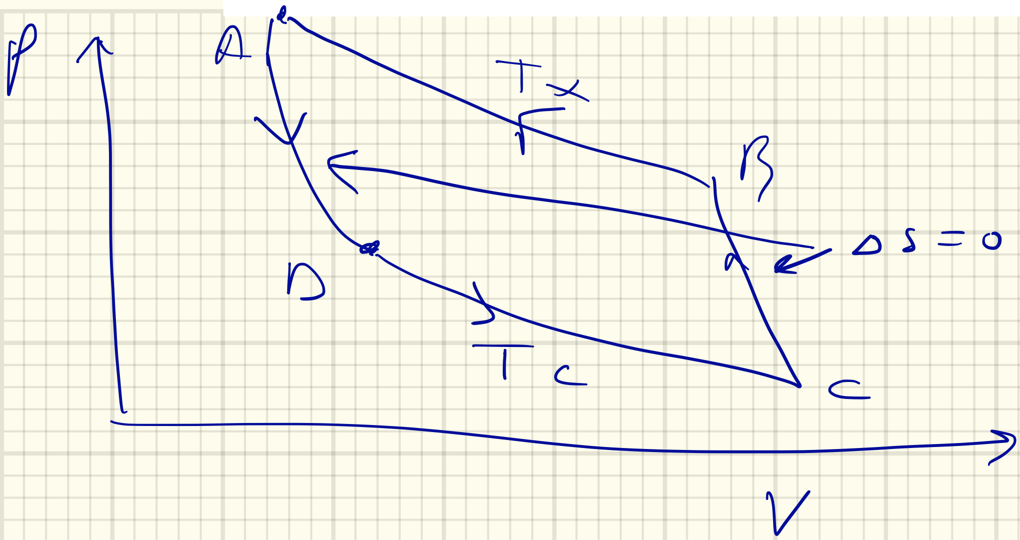
$= (1-\gamma) P_b V_b \left(\frac{V_b}{V}\right)^{\gamma-1} \Big|_{V_b}^{V_c} = (1-\gamma) N T_H \left\{ 2^{1-\gamma} - 1 \right\}$

d) $P_b V_b = P_c V_c \left(\frac{V_c}{V_b}\right)^{\gamma-1}$

$T_c = T_b \cdot 2^{1-\gamma}$, $T_x = T_H \cdot 2^{1-\gamma}$

$\text{eff} = 1 - \frac{T_x}{T_H} = 1 - 2^{1-\gamma}$

5. Consider a refrigerator built by an inverse Carnot cycle. What is the efficiency of the refrigerator in terms of the temperatures T_C and T_X?



$$\text{eff} = \frac{Q_{cd}}{W} = \frac{T_c \Delta S}{T_x \Delta S - T_c \Delta S}$$

$$= \frac{T_c}{T_x - T_c}$$, can be > 1 !

6. Consider a hydrodynamic slab which has a Gaussian profile along the x direction but is translationally invariant in the y and z directions. Assume the matter behaves as an ideal gas of non-relativistic particles. Initially, the matter is at rest and has a profile,

$$\rho(x, t=0) = \rho_0 \exp(-x^2/2R_0^2),$$

with an initial uniform temperature T_0. Assume that as it expands it maintains a Gaussian profile with a Gaussian radius $R(t)$.

(a) Show that entropy conservation requires

$$T(t) = T_0 \left(\frac{R_0}{R}\right)^{2/3}.$$

(b) Assuming the velocity has the form $v = A(t)x$, show that conservation of particle current gives

$$A = \frac{\dot{R}}{R}.$$

(c) Show that the hydrodynamic expression for acceleration gives

$$\dot{A} + A^2 = \frac{R_0^{2/3} T_0}{m R^{8/3}}$$

Putting these two expressions together show that $R(t)$ can be found by solving and inverting the integral,

$$t = \sqrt{\frac{m}{3T_0}} \int_{R_0}^{R} \frac{dx}{\sqrt{1-(R_0/x)^{2/3}}}.$$

a) $\frac{S}{N} = \frac{5}{2} + \ln\left[\frac{(mT)^{3/2}}{\rho(2\pi\hbar^2)^{3/2}}\right]$ at given ρ

$= C + \ln(T^{3/2}/\rho)$ at given ρ

$\frac{S}{N} = \int dx\, \rho\left[C + \ln(T^{3/2}/\rho)\right] \Big/ \int dx\, \rho$; $\rho = \frac{e^{-x^2/2R^2}}{R}$

$= C + \frac{3}{2}\ln T + \frac{\int dx\, \rho\left[\ln R + \frac{x^2}{2R^2}\right]}{\int dx\, \rho}$

$= C + \frac{1}{2} + \frac{3}{2}\ln T + \ln R$

$T^{3/2} R = \text{const.}$

$T R^{2/3} = T_0 R_0^{2/3}$

$T = T_0 \left(\frac{R_0}{R}\right)^{2/3}$

7. Consider a gas obeying the Van der Waals equation of state,

$$P = \frac{\rho T}{1-\rho/\rho_s} - a\rho^2.$$

78

(a) First, using Maxwell relations, show that the speed of sound for $P(\rho, T)$ is given by

$$mc_s^2 = \left.\frac{\partial P}{\partial \rho}\right|_T + \left(\left.\frac{\partial P}{\partial T}\right|_\rho\right)^2 \frac{T}{\rho^2 C_V}.$$

(b) Consider a gas described by the Van der Waals equation of state, which has a density equal to that of the critical point, $\rho_s/3$. What is the range of temperatures for which the matter has unstable sonic modes? Express your answer in terms of the critical temperature T_c.

$$\delta P = \left.\frac{\partial P}{\partial \rho}\right|_T \delta\rho + \left.\frac{\partial P}{\partial T}\right|_\rho \delta T$$

$$\delta S = \left.\frac{\partial S}{\partial \rho}\right|_T \delta\rho + \left.\frac{\partial S}{\partial T}\right|_\rho \delta T = 0$$

$$\delta P|_S = \delta\rho \left\{ \left.\frac{\partial P}{\partial \rho}\right|_T - \frac{\left.\frac{\partial P}{\partial T}\right|_\rho \left.\frac{\partial S}{\partial \rho}\right|_T}{\left.\frac{\partial S}{\partial T}\right|_\rho} \right\}$$

$$dE = TdS - PdV$$

$$d(E-TS) = -SdT - PdV$$

$$\left.\frac{\partial P}{\partial T}\right|_\rho = \left.\frac{\partial S}{\partial V}\right|_T = -\rho^2 \left.\frac{\partial S}{\partial \rho}\right|_T$$

$$\delta P\Big|_S = \delta\rho \left\{ \left.\frac{\partial P}{\partial \rho}\right|_T + \frac{\left(\frac{\partial P}{\partial T}\right)^2 1/\rho^2}{\left(\frac{\partial S}{\partial T}\right)} \right\}$$

$$C_V = T\left.\frac{\partial S}{\partial T}\right|_\rho$$

$$mc_s^2 = \left.\frac{\partial P}{\partial \rho}\right|_S = \left.\frac{\partial P}{\partial \rho}\right|_T + \frac{\left(\left.\frac{\partial P}{\partial T}\right|_\rho\right)^2}{C_V} \frac{T}{\rho^2}$$ ✓

$$mc_s^2 = \frac{T}{1-\rho/\rho_s} + \frac{\rho T/\rho_s}{(1-\rho/\rho_s)^2} - 2a\rho + \frac{\rho^2}{(1-\rho/\rho_s)^2}\frac{2}{3}\frac{T}{\rho^2}$$

if $\rho/\rho_s = 1/3$

$$mc_s^2 = \frac{3}{2}T + \frac{1}{3}\left(\frac{3}{2}\right)^2 T + \frac{2}{3}\left(\frac{3}{2}\right)^2 T - 2a\rho$$

$$= \frac{15}{4}T - \frac{2}{3}a\rho_s \to 0 \text{ when } T = \frac{8}{45}a\rho_s$$

$T_c = 8/27\,\rho_s$, unstable for $0 < T < \frac{8}{45}a\rho_s$

b) $$\frac{\partial \rho}{\partial t} + v\frac{\partial \rho}{\partial x} + \rho\frac{\partial v}{\partial x} = 0$$

$$v = A(t)\,x\,,\quad \rho = e^{-x^2/2R^2}/R$$

$$\underbrace{-\frac{\dot R}{R}}_{\text{}} + \frac{x^2}{R^3}\dot R - A x\frac{x}{R^2} + A = 0$$

$$A = \dot R/R \qquad\qquad A = \dot R/R$$

c) $$m\rho\frac{Dv}{Dt} = -T\,\partial_x \rho = \left(\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x}\right)m\rho$$

$$T\frac{x}{R^2} = \left(\dot A x + A x A\right)m$$

from a) $$T = T_0\left(\frac{R_0}{R}\right)^{2/3}$$

$$\frac{T_0}{m}\,\frac{R_0^{2/3}}{R^{8/3}} = \dot A + A^2 \quad ✓$$

$$\frac{T_0}{m}\,\frac{R_0^{2/3}}{R^{8/3}} = \frac{\ddot R}{R} - \cancel{\frac{\dot R^2}{R^2}} + \cancel{\frac{\dot R^2}{R^2}}$$

$$m\ddot R = T_0\,\frac{R_0^{2/3}}{R^{5/3}}$$

acts like "potential"

$$V(R) = T_0\,\frac{R_0^{2/3}}{R^{2/3}}\cdot\frac{3}{2}$$

$$\frac{1}{2}m\dot R^2 = \frac{3}{2}T_0\left(1-\left(\frac{R_0}{R}\right)^{2/3}\right)$$

$$\frac{dR}{dt} = \sqrt{\frac{2}{m}\frac{3}{2}T_0\left(1-\left(\frac{R_0}{R}\right)^{2/3}\right)}$$

$$t = \int_{R_0}^{R}\frac{dR'}{\sqrt{\frac{3T_0}{m}\left(1-\left(\frac{R_0}{R}\right)^{2/3}\right)}} \quad ✓$$

8. Consider an initially thermalized three-dimensional Gaussian distribution for the phase space density of non-relativistic particles of mass m,

$$f(\mathrm{p}, \mathrm{r}, t=0) = f_0 \exp\left\{-\frac{r^2}{2R_0^2} - \frac{p^2}{2mT_0}\right\}.$$

Assume the particles move freely for $t > 0$.

(a) At a given r and $t > 0$, show that $f(\mathrm{p}, \mathrm{r}, t)$ can be expressed in terms of a locally thermalized distribution of the form,

$$f(\mathrm{p}, \mathrm{r}, t) = C(t) e^{-r^2/2R^2(t)} \exp\left\{-\frac{(\mathrm{p} - m\mathrm{v}(\mathrm{r}, t))^2}{2mT(\mathrm{r}, t)}\right\}.$$

Then, find $C(t)$, $R(t)$, $\mathrm{v}(\mathrm{r}, t)$ and $T(\mathrm{r}, t)$. In addition to r and t, these parameters should depend on the initial Gaussian size R_0, the initial temperature T_0 and the mass m.

(b) Find the density as a function of r and t, then compare your result for the density and temperature to that for a hydrodynamic expansion of the same initial distribution as described in the example given in the lecture notes.

(c) Using the fact that a hydrodynamic expansion assumes infinitely high collision rate, and that the Boltzmann solution was for zero collision rate, make profound remarks about how the hydrodynamic and free-streaming evolutions compare with one another.

(d) Calculate the total entropy as a function of time for the previous problem assuming f_0 is small.

a) $$f(p, r, t>0) = f_0 \exp\left\{-\frac{(\vec r - \vec v_p t)^2}{2R_0^2} - \frac{p^2}{2mT_0}\right\}$$

$$= f_0 \exp\left\{-\frac{r^2}{2R_0^2} - \frac{p^2}{2mT_0} - \frac{p^2 t^2}{2R_0^2 m^2} + \frac{\vec p \cdot \vec r}{R_0^2 m} t\right\}$$

$$= f_0 \exp\left\{\frac{-r^2}{2R_0^2} - \frac{1}{2mT}\left(\vec p - \frac{\vec r t}{R_0^2} T\right)^2 + \frac{r^2 T t^2}{2mR_0^2}\right\}$$

$$\frac{1}{T} = \frac{1}{T_0} + \frac{t^2}{mR_0^2}, \quad T = \frac{1}{\frac{1}{T_0} + t^2/mR_0^2} = \frac{T_0}{1 + \frac{T_0 t^2}{mR_0^2}}$$

$$f = f_0 \exp\left\{\frac{-r^2}{2R^2} - \frac{1}{2mT}(\vec p - m\vec v)^2\right\}$$

$$T = \frac{T_0}{1 + T_0 t^2/mR_0^2}, \qquad \vec v = \frac{\vec r T t}{R_0^2 m}$$

$$\frac{1}{R^2} = \frac{1}{R_0^2} - \frac{T t^2}{mR_0^4}, \quad R^2 = \frac{1}{\frac{1}{R_0^2} - \frac{T t^2}{mR_0^4}} = \frac{R_0^2}{1 - \frac{T}{mR_0^2} t^2}$$

$$R^2 = \frac{R_0^2}{1 - \frac{t^2}{mR_0^2} T_0 \frac{1}{1 + T_0 t^2/mR_0^2}} = \frac{R_0^2(1 + T_0 t^2/mR_0^2)}{1 + T_0 t^2/mR_0^2 - \frac{T_0 t^2}{mR_0^2}} = R_0^2 + \frac{T_0 t^2}{m} = R^2$$

c) $\rho = \frac{1}{(2\pi\hbar)^3}\int f\, d^3p$ $\quad R^2 = R_0^2 + \frac{T_0 t^2}{m}$

$$= \frac{(mT)^{3/2}}{\hbar^3(2\pi)^{3/2}} f_0\, e^{-r^2/R^2} = \text{same as hydro}$$

$$T = T_0 \frac{1}{1 + T_0 \frac{t^2}{mR_0^2}} = \frac{T_0 R_0^2}{R^2}$$

$$\vec{v}_{coll} = \frac{T t}{R_0^2 m}\vec{r} = \text{same as hydro}$$

$$= \frac{T_0 t}{R^2 m}\vec{r} = \text{same as hydro}$$

Because free-streaming solution maintains form with local thermal equilibrium,

$$f \sim e^{-(\vec{p} - m\vec{v}(\vec{r}))^2/2mT},$$

collisions do nothing. – because collisions can only make distribution more thermal.

thus zero collisions = same as ∞ collisions

d) $$S = \int \frac{d^3r\, d^3p}{(2\pi\hbar)^3} f(1 - \ln f)$$

$$= N - \int \frac{d^3p\, d^3r}{(2\pi\hbar)^3} f\left[\ln f_0 - \frac{r^2}{2R^2} - \frac{(p - mV)^2}{2mT}\right]$$

$$= 4N - N\ln f_0$$ independent of time!

9. Assume that there exists a massive species of neutrinos, $m_\nu = 10$ eV. Further assume that it froze out at the same time as the example of the text, when the Hubble time as 10^7 years and the temperature was 4000 K. If the Hubble expansion was without acceleration after that point, and if the current Hubble time is 14×10^9 years, find:

 (a) the current effective temperature of the massive neutrino.

 (b) If the neutrino has two polarizations (just like photons) what would be the relative population, N_ν/N_γ, at freezeout? Assume the chemical potential for the neutrino is zero (otherwise there would be more neutrinos than anti-neutrinos) and treat the neutrino non-relativistically,

$$f_\nu(p) = \frac{e^{-\beta\epsilon}}{1+e^{-\beta\epsilon}} \approx e^{-\beta(m+p^2/2m)}.$$

$m \gg T$

a) $f = e^{-E_0/T}$

$$E_0 = m + \frac{p_0^2}{2m} = m + \frac{p^2}{2m}\frac{a^2}{a_0^2}$$

$$f = e^{-\frac{m}{T_0} - \frac{p^2}{m(T \cdot a_0^2/a^2)}}$$

$$= e^{-\frac{m}{T} - \frac{p^2}{2mT} - m\left(\frac{1}{T_0} - \frac{1}{T}\right)}$$

$$T_{eff} = T\, a_0^2/a^2, \quad \mu_{eff} = m\left(1 - \frac{T}{T_0}\right)$$

b) $$\rho_\gamma = \frac{2}{(2\pi\hbar)^3}\,\zeta(3)\int 4\pi p^2 dp\, e^{-p/T_0}$$

$\zeta(3)$ accounts for $e^{-p/T} \to \frac{e^{-p/T}}{1-e^{-p/T}}$

$$= \frac{2\,T_0^3}{\pi^2\hbar^3}\,\zeta(3)$$

$$\rho_\nu = \frac{2}{(2\pi\hbar)^3}\int 4\pi p^2 dp\; e^{-p^2/2mT_0 - m/T_0}$$

$$= e^{-m/T_0}\cdot 2\,\frac{(mT)^{3/2}}{(2\pi\hbar^2)^{3/2}}$$

$$\frac{N_\nu}{N_\gamma} = \left(\frac{m}{T}\right)^{3/2}\frac{\pi^{1/2}}{\zeta(3)}$$

10. Consider a hot nucleus of radius 5 fm at a temperature of 1 MeV. The chemical potential for a cold (or warm) nucleus is approximately the binding energy per particle, $\mu \approx -7$ MeV. Estimate the mean time between emitted neutrons. (Treat the nucleus as if it has fixed temperature and assume a Boltzmann distribution (not Fermi) for $f(\mathbf{p})$).

$$\frac{d\Gamma}{d^3p} = \overset{\text{spins}}{2} e^{\beta\mu} e^{-E/T} \frac{v}{(2\pi\hbar)^3} \pi R^2 , \quad E = p^2/2mT$$

$$\Gamma = 2 e^{\beta\mu} \pi R^2 \int \frac{4\pi p^2 dp}{\hbar} \frac{p}{m} e^{-p^2/2mT}$$

$$= 8 e^{\beta\mu} \pi^2 R^2 \int u\,du\, e^{-u} \cdot T^2 \cdot 2m$$

$$= 16 e^{\beta\mu} \frac{\pi^2 R^2 T^2}{(2\pi\hbar)^3} m = 2 \frac{m R^2 T^2}{\pi \hbar^3} e^{\beta\mu}$$

$$\hbar c = 197.326 \text{ MeV fm}$$

$$\Gamma = 2 \cdot \frac{938.3 \cdot 25}{\pi \; (197.326)^3} \exp(-7) \quad (\text{fm}/c)^{-1}$$

$$= 1.772 \cdot 10^{-6} \; (\text{fm}/c)^{-1}$$

$$= 1.772 \cdot 10^{-6} \; (3 \cdot 10^8 \cdot 10^{15}) \quad s^{-1}$$

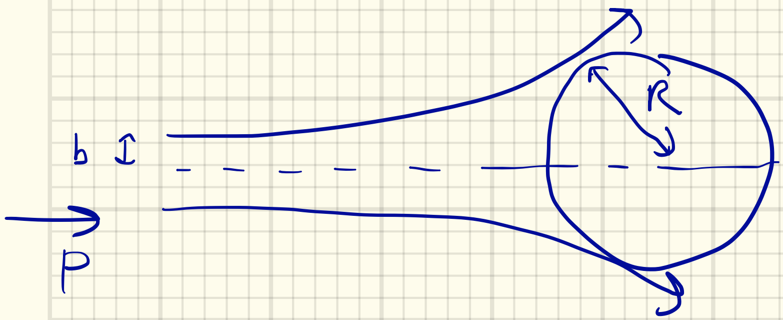
$$= 5.32 \cdot 10^{4} \quad \text{decays/s}$$

$$\tau = 1/\Gamma = 1.88 \cdot 10^{-12} \; s$$

11. Consider a hot nucleus of radius R with an electric charge of Z at temperature T. Assuming that protons and neutrons have the same chemical potential, find the ratio of proton spectra to neutron spectra,

$$\frac{dN_{\rm prot}/d^3p\,dt}{dN_{\rm neut}/d^3p\,dt}$$

as a function of the momentum p. Approximate the two masses as being equal, $m_n = m_p$, and neglect quantum degeneracy. Assume that all incoming nucleons would be captured and thermalized if they reach the position R. HINT: The emission ratio equals the ratio of capture cross sections.



$$\sigma_{neut} = \pi R^2$$

$$\sigma_{prot} = \pi b^2$$

$$L_z = L_z$$

$$pb = \sqrt{p^2 - 2m\frac{Ze^2}{R}} \cdot R$$

$$\frac{\pi b^2}{\pi R^2} = \frac{p^2 - 2mZe^2/R}{p^2}$$

$$= 1 - \frac{2mZe^2}{p^2 R}$$

$$= \frac{dN_{prot}/d^3p}{dN_{neut}/d^3p}$$

12. A drift detector works by moving the electrons ionized by a track through a gas towards readout plates.

(a) Assuming the mean free path of the electrons is $\lambda = 300$ nm, and assuming their velocity is thermal at room temperature ($v_{\text{therm}} \approx \sqrt{T/m}$), estimate the size R (Gaussian radius) that the diffusion cloud imprints onto the plates after drifting for 200 μs. Use the approximation that $D \approx \lambda v_{\text{therm}}$.

(b) Assume an electric field E is responsible for the drift velocity. If the drift velocity is approximately $a \cdot \tau/2$, where a is the acceleration and $\tau = \lambda/v_{\text{therm}}$ is the collision time, find an analytic expression for the Gaussian size R of the cloud if it travels a distance L. Give R in terms of the temperature T, the electron mass m and charge e, the electric field E, the mean free path λ and L.

a) $$R^2 = 2Dt = 2\lambda\sqrt{T/m}\cdot t$$

$$T = 300\,K = 4.14\cdot 10^{-21}\,J$$

$$m = 9.11\cdot 10^{-31}\,kg$$

$$v_{therm} = \sqrt{\frac{T}{m}} = 6740\,m/s$$

$$R = \sqrt{2\cdot 300\cdot 10^{-9}\; 6740\cdot 200\cdot 10^{-6}}\;m$$

$$= 9.0\cdot 10^{-4}\,m = 90\,\mu m$$

b) $$a = eE,\quad \Delta x = \frac{1}{2}a\tau^2 = \frac{1}{2}a\left(\frac{\lambda}{v_{therm}}\right)^2$$

$$v_{drift} = \frac{\Delta x}{\tau} = \frac{1}{2}a\left(\frac{\lambda}{v_{therm}}\right)^2\cdot\frac{v_{therm}}{\lambda}$$

$$= \frac{1}{2}a\frac{\lambda}{v_{therm}}$$

$$t_{drift} = L/v_{drift} = \frac{2Lv_{therm}}{a\lambda}$$

$$R^2 = 2D\,t_{drift} = 2\lambda\sqrt{T/m}\cdot\frac{2L\sqrt{T/m}}{a\lambda}$$

$$= 4\frac{T}{m}\frac{L}{a} = 4\frac{LT}{emE}$$

13. A cloud of radioactive mosquitos is being blown by the wind parallel to a high-voltage mosquito-zapping plate. At at time $t = 0$, the cloud is at the first edge of the plate ($x = 0$). The probability they start out at a distance y from the plate is:

$$\rho(y, t = 0) = A_0 y e^{-y^2/(2R_0^2)}$$

The speed of the breeze is v_x, and the length of the plate is L. The mosquito's motion in the y direction can be considered a diffusion process with diffusion constant D.

(a) What is the distribution $\rho(y, t)$? Assume that the form for $\rho(t)$ is the same as for ρ_0 only with A and R becoming functions of t. Solve only for $t < L/v_x$.

(b) What fraction of mosquitos survive?

(a) $\rho = A(t)\, y\, e^{-y^2/2R^2(t)}$

$$\dot\rho = \left[\frac{\dot A}{A} + \frac{y^2}{R^3}\dot R\right]\rho$$

$$\partial_y^2 \rho = \left[\frac{-1}{R^2} - \frac{2}{R^2} + \frac{y^2}{R^4}\right]\rho$$

$$\frac{\dot A}{A} = \frac{-3D}{R^2}, \quad \frac{\dot R}{R^3} = \frac{D}{R^4} \qquad \dot R = \frac{D}{R}$$

$$\boxed{\frac{1}{2}\frac{d}{dt}R^2 = D \quad R^2 = 2Dt \quad R = \sqrt{2Dt}} \qquad \boxed{\dot A = A\left(-\frac{3D}{2Dt}\right) = -\frac{3}{2}\frac{A}{t} \quad A = C t^{-3/2}}$$

$$\rho = C t^{-3/2} e^{-y^2/4Dt}$$

translate by time t_0

$$\rho \propto C (t + t_0)^{-3/2} e^{-y^2/4D(t+t_0)}$$

$$2Dt_0 = R_0^2$$

$$\rho \propto C' \frac{y}{(R_0^2 + 2Dt)^{3/2}} e^{-y^2/2(R_0^2 + 2Dt)}$$

(b) $$\int \rho\, dy = \frac{C'}{(R_0^2 + 2Dt)^{3/2}} (R_0^2 + 2Dt)$$

$$= C'(R_0^2 + 2Dt)^{-1/2}$$

survival frac $= \dfrac{1}{\sqrt{1 + 2Dt/R_0^2}}$

14. For an expanding system, the Diffusion equation is modified by adding an extra term proportional to $\nabla \cdot \mathrm{v}$,

$$\frac{\partial \rho}{\partial \tau} + \rho(\nabla \cdot \mathrm{v}) = D\nabla^2\rho.$$

The second term accounts for the fact that the density would fall due to the fact that the matter is expanding, even if the the particles were not diffusing. For a Hubble expansion,

$\mathrm{v} = \mathrm{r}/\tau$, and the $\nabla \cdot \mathrm{v} = 3/\tau$ (would be $2/\tau$ or $1/\tau$ in 2-D or 1-D). Thus in a Hubble expansion, the diffusion equation becomes

$$\frac{\partial \rho}{\partial t} + \frac{3}{\tau}\rho = D\nabla^2\rho.$$

Instead of the position $\mathbf{r}$, one can use the variable

$$\vec{\eta} \equiv \frac{\vec{r}}{\tau}.$$

The advantage of using η is that because the velocity gradient is $1/\tau$, the velocity difference between two points separated by $dr = \tau d\eta$ is $dv = d\eta$. Thus if two particles move with the velocity of the local matter, their separation $\eta_1 - \eta_2$ will remain fixed. Next, one can replace the density $\rho = dN/d^3r$ with

$$\rho_\eta \equiv \frac{dN}{d^3\eta} = \tau^3\rho.$$

Here, we have been a rather sloppy with relativistic effects, but for $|\eta|$ much smaller than the speed of light, they can be ignored. One can now rewrite the diffusion equation for ρ_η,

$$\frac{\partial \rho_\eta}{\partial \tau} = D\nabla^2\rho_\eta.$$

This looks like the simple diffusion equation without expansion, however because the density is changing D is no longer a constant which invalidates using the simple Gaussian solutions discussed in the chapter. For an ultrarelativistic gas, with perturbative interactions, the scattering cross sections are roughly proportional to $1/T^2$, and the density falls as $1/T^3$, which after considering the fact that the temperature then falls as $1/\tau$, the diffusion constant would roughly rise inversely with the time,

$$D(\tau) = D_0\frac{\tau}{\tau_0}.$$

This time dependence would be different if the particles had fixed cross sections, or if the gas was not ultra-relativistic. However, we will assume this form for the questions below.

(a) Transform the three-dimensional diffusion equation,

$$\frac{\partial \rho_\eta}{\partial \tau} = D(\tau)\nabla^2\rho_\eta$$

into an equation where all derivatives w.r.t. r are replaced with derivatives w.r.t. η.

(b) Rewrite the expression so that all mention of τ is replaced by $s \equiv \ln(\tau/\tau_0)$.

(c) If a particle is at $\vec{\eta} = 0$ at τ_0, find $\rho_\eta(\eta, s)$.

a) $\dfrac{\partial \rho}{\partial \tau} = D\nabla^2\rho$, $\rho = \rho_\eta/\tau^3$

$\vec{\eta} = \vec{r}/\tau$

$\nabla^2\rho = \dfrac{1}{\tau^2}\nabla_\eta^2\rho = \dfrac{1}{\tau^5}\nabla_\eta^2\rho_\eta$

$\partial_\tau\left(\dfrac{\rho_\eta}{\tau^3}\right) = -\dfrac{3}{\tau^4}\rho_\eta + \dfrac{1}{\tau^3}\partial_{\cancel{s}}\rho_\eta = D\dfrac{\nabla_\eta^2}{\tau^5}\rho_\eta$

$+\dfrac{3}{\tau^4}\rho_\eta$

$\partial_s\rho_\eta = D\dfrac{1}{\tau^2}\nabla_\eta^2\rho_\eta$

15. Consider Fick's law for the number density and the number current, $\vec{j} = -\boldsymbol{D}\nabla\rho$.

(a) Rewrite Fick's law in terms of the gradient of the chemical potential, showing that $\boldsymbol{D}$ is replaced by $\boldsymbol{D}\chi$ where $\chi = \partial\rho/\partial\mu$.

(b) Replacing the gradient of the chemical potential with the gradient of electric potential, assuming the particles have charge e, find an expression for the electric current, $\vec{j}_e = e\vec{j}$, in terms of a gradient of the electric potential energy $e\Phi$.

(c) Express the electric conductivity, σ, in terms of D, χ and e.

a) $\vec{J} = -D(\vec{\nabla}\mu)\cdot\frac{\partial\rho}{\partial\mu} = -D\chi\nabla\mu$

b) $\mu = \text{energy} = -eEX = e\Phi$

$\vec{\nabla}\mu = eE$

$\vec{J} = -D\chi eE$

number current

$\vec{J}_e = -De^2\chi E$

charge current

c) $\vec{J}_e = \sigma\vec{E}$

$\sigma = De^2\chi$

16 ~~15~~. Consider Eq. (4.108) for a one-dimensional system:

(a) In terms of γ, T, Δt and m, estimate the amount of time required for the variance of the sum of random impulses in one direction to reach mT, the thermal variance.

(b) Calculate $\langle v(t=0)v(t)\rangle$, the velocity-velocity correlation in the limit $\Delta t \to 0$.

(c) Calculate the r.m.s. distance traveled by a particle in time t in the same limit.

(d) For large times, $t >> 1/\gamma$, estimate the diffusion constant.

4.108
$$T = \frac{\sigma^2}{2m\gamma\Delta t}.$$

$$N_{pulses}\,\sigma^2 = mT$$

a) $$\frac{t}{\Delta t}\sigma^2 = mT,$$

$$t = \frac{mT\Delta t}{\sigma^2} = \frac{1}{2\gamma}$$

b) $$\langle v(t=0)\,v(t=0)\rangle = \frac{2T}{m}$$

$$\frac{d}{dt}\langle v(t=0)\,v(t)\rangle = -\gamma\langle v(t)\,v(t)\rangle$$

random changes don't contribute to $\langle\ldots\rangle$

$$\langle v(0)\,v(t)\rangle = \frac{2T}{m}e^{-\gamma t}$$

c) $$v(t) = v_0 e^{-\gamma t} + \sum_{n=1}^{N=t/\Delta t} \Delta v_n(t)\, e^{-\gamma(t-n\Delta t)}$$

$$x(t) = \frac{v_0}{\gamma}\left(1-e^{-\gamma t}\right) + \sum_n \frac{\Delta v_n}{m\gamma}\left(1-e^{-\gamma(t-n\Delta t)}\right)$$

$$\langle x(t)^2\rangle = \frac{v_0^2}{\gamma^2}\left(1-e^{-\gamma t}\right)^2 + \sum_n \frac{\sigma^2}{m^2\gamma^2}\left(1-e^{-\gamma(t-n\Delta t)}\right)^2$$

$$= \frac{v_0^2}{\gamma^2}\left(1-e^{-\gamma t}\right)^2 + \sum_n \frac{\sigma^2}{m^2\gamma^2\Delta t}\Delta t\left(1-2e^{-\gamma(t-n\Delta t)}+e^{-2\gamma(t-n\Delta t)}\right)$$

$$= \frac{T}{m\gamma^2}\left(1-e^{-\gamma t}\right)^2 + \frac{T}{2m\gamma}\int dt\left(1-2e^{-\gamma(t-n\Delta t)}+e^{-2\gamma(t-n\Delta t)}\right)$$

$$= \frac{T}{m\gamma^2}\left(1-e^{-\gamma t}\right)^2 + \frac{T}{2m\gamma}\left[t-\frac{2}{\gamma}\left(1-e^{-\gamma t}\right)+\frac{1}{2\gamma}\left(1-e^{-2\gamma t}\right)\right]$$

As $t\to\infty$ $$= \frac{T}{m\gamma^2} + \frac{T}{2m\gamma}t - \frac{T}{m\gamma^2} + \frac{T}{4m\gamma^2} = \frac{Tt}{2m\gamma} + \frac{T}{4m\gamma^2}$$

5 Chapter 5

5.5 Problems

1. The speed of sound in copper is 3400 m/s, and the number density is $\rho_{Cu} = 8.34 \times 10^{28}$ m^{-3}.

   (a) Assuming there are two free electrons per atom, find an expression for C_V/N_a (where N_a is the number of atoms) from the free electrons in copper by assuming a free gas of electrons. Use the expression from chapter 2 for a low-T Fermi gas,

   $$\delta E = T^2 \frac{\pi^2}{6} D(\epsilon),$$

   where $D(\epsilon)$ is the density of single particle electron states at the Fermi surface. Give answer in terms of T and the Fermi energy ϵ_F.

   (b) In terms of ϵ_F and $\hbar\omega_D$, find an expression for the temperature at which the specific heat from electronic excitations equals that from phonons. Use the low T expression for the specific heat of phonons.

   (c) What is $\hbar\omega_D$ in eV? in K?

   (d) What is ϵ_F in eV? in K?

   (e) What is the numerical value for the answer in (b) in K?

a) $$\delta E = \frac{\pi^2}{6} D(\varepsilon) T^2$$

$$D(\varepsilon) = \frac{2V}{(2\pi\hbar)^3} 4\pi p^2 \frac{dp}{d\varepsilon} = \frac{Vmp}{\pi^2\hbar^3}$$

$$\delta E = V\frac{mp_f T^2}{6\hbar^3}$$

$$C_V = \frac{dE}{dT} = \frac{Vmp_f T}{3\hbar^3}$$

$$N_a = \#\text{ atoms} = \frac{1}{2}\rho_e V$$

$$= \frac{1}{(2\pi\hbar)^3} V \frac{4\pi}{5} p_f^5$$

$$\frac{C_V}{N_a} = \frac{Vmp_f T}{3\hbar^3} \frac{(2\pi\hbar)^3}{\frac{4\pi}{3}p_f^3 \cdot V} = \frac{2\pi^2 mT}{p_f^2}$$

$$= \pi^2\frac{T}{\varepsilon_F}$$

b) $$C_V^{(\text{phonons})} = 3\frac{d}{dT} V \frac{4\pi c_s}{(2\pi\hbar)^3}\int p^3 dp\, e^{-p c/T}\ \zeta(4)$$

$$= \frac{3c_s}{2\hbar^3} V \zeta(4)\cdot 3!\ \frac{d}{dT}\frac{T^4}{c_s^4}$$

$$= \frac{36V}{(\hbar c_s)^3} T^3 \zeta(4) = \frac{Vmp_f T}{3\hbar^3}$$

$$T^2 = \frac{mp_f\pi^2 c_s^3}{108\,\zeta(4)}$$

5.5 Problems

1. The speed of sound in copper is 3400 m/s, and the number density is $\rho_{\mathrm{Cu}} = 8.34 \times 10^{28}$ m^{-3}.

   (a) Assuming there are two free electrons per atom, find an expression for C_V/N_a (where N_a is the number of atoms) from the free electrons in copper by assuming a free gas of electrons. Use the expression from chapter 2 for a low-T Fermi gas,

   $$\delta E = T^2 \frac{\pi^2}{6} D(\epsilon),$$

   where $D(\epsilon)$ is the density of single particle electron states at the Fermi surface. Give answer in terms of T and the Fermi energy ϵ_F.

   (b) In terms of ϵ_F and $\hbar\omega_D$, find an expression for the temperature at which the specific heat from electronic excitations equals that from phonons. Use the low T expression for the specific heat of phonons.

   (c) What is $\hbar\omega_D$ in eV? in K?

   (d) What is ϵ_F in eV? in K?

   (e) What is the numerical value for the answer in (b) in K?

b) continued

$$T^2 = \frac{m p_f \pi^2 c_s^3}{108\, \zeta(4)}$$

From Eq 5.6, $c_s^3 = \omega_D^3 \left(\frac{V}{6\pi^2 N_a}\right) = \frac{(\omega_D \hbar)^3}{12\pi^2 \varepsilon_F}$

$$T^2 = \frac{m p_f \pi^2}{108}\left(\frac{\omega_D \hbar}{p_f}\right)^3 \frac{\pi^4}{90} = \frac{5 m (\omega \hbar)^3}{6\pi^2 p_f^2} = \frac{5}{12}\frac{(\hbar\omega_D)^3}{\pi^2 \varepsilon_F}$$

$$T = \sqrt{\frac{5(\hbar\omega_D)^3}{12\pi^2 \varepsilon_F}}$$

c) $\omega_D = c_s \left(\frac{6\pi^2 N_a}{V}\right)^{1/3} = c_s (6\pi^2 \rho_{cu})^{1/3}$

$= 5.79 \cdot 10^{13}\ s^{-1}$

$\hbar\omega_D = 0.038\ eV = 442\ K$

d) $\varepsilon_F = \frac{p_f^2}{2m}$, $p_f = (3\pi^2\hbar^3 \rho_e)^{1/3} = (6\pi^2\hbar^3\rho_{cu})^{1/3}$

$\varepsilon_F = \frac{p_f^2}{2m} = \frac{(6\pi^2\hbar^3\rho_{cu})^{1/3}}{2m} = 11.05\ eV = 1.28 \cdot 10^5 K$

e) $T = \sqrt{\frac{5(\hbar\omega_D)^3}{12\pi^2 \varepsilon_F}} = 5.3\ K$

2. Consider the mean-field solution to the Ising model of Eq. (5.18).

(a) For small temperatures, show that the variation $\delta\langle\sigma\rangle = 1 - \langle\sigma\rangle$ is:

$$\delta\langle\sigma\rangle \approx 2e^{-2qJ/T}.$$

(b) Consider a function of the form, $y(T) = e^{-1/T}$. Find dy/dT, d^2y/dT^2, and d^ny/dT^n evaluated at $T = 0^+$. What does this tell you about doing a Taylor expansion of $y(T)$ about $T = 0^+$?

a) $\langle \sigma \rangle = \tanh(\beta q J \langle \sigma \rangle)$

$$1 - \delta\sigma = \frac{1 - e^{-2\beta q J(1-\delta\sigma)}}{1 + e^{-2\beta q J(1-\delta\sigma)}}$$

$$\cong 1 - 2e^{-2\beta q J(1-\delta\sigma)}$$

$$\delta\sigma \cong 2e^{-\beta q J}$$

b) $y = e^{-1/T}$, $\frac{dy}{dt} = \frac{1}{T^2} e^{-1/T} \to \infty$ as $T \to 0$

$$\frac{d^2y}{dT^2} = \infty$$

can't do Taylor expansion about $T = 0$!
can't expand functions in powers of T
if there are $e^{-1/T}$ terms!

3. Show that in the mean field approximation of the Ising model the susceptibility,

$$\chi \equiv \frac{d\langle\sigma\rangle}{dB},$$

becomes

$$\chi = \frac{(1-\langle\sigma\rangle^2)\mu}{T - T_c + \langle\sigma\rangle^2 T_c}.$$

$$\sigma = \tanh\left(\beta(q J \sigma + \mu B)\right)$$

$$d\sigma = (1 - \tanh^2)\left[\beta q J \, d\sigma + \beta \mu \, dB\right]$$

$$d\sigma\left(1 - \beta q J(1-\sigma^2)\right) = (1-\sigma^2)\beta\mu \, dB$$

$$\chi = \beta\mu \frac{1-\sigma^2}{1 - \beta q J + \beta q J \sigma^2}$$

$$= \mu \frac{(1-\sigma^2)}{T - T_c + \sigma^2 T_c}$$

$$T - T_c + \langle\sigma\rangle^2 T_c$$

4. The total energy for the Ising model in the mean field approximation from summing over all the sites in Eq. (5.16) is

$$H = -\frac{N}{2} qJ\langle\sigma\rangle^2 - N\mu B\langle\sigma\rangle,$$

where N is the number of sites, and the factor of $1/2$ is a correction for double counting. In terms of $T, T_c, \mu B$ and $\langle\sigma\rangle$, find an expression for

$$C_v = \frac{1}{N}\frac{dE}{dT}.$$

$$dE = (-NqJ\sigma - N\mu B)\, d\sigma$$

$$\sigma = \tanh \beta(qJ\sigma + \mu B)$$

$$d\sigma = (1-\sigma^2)\left[\beta q J\, d\sigma + (qJ\sigma + \mu B)\, d\beta\right]$$

$$d\sigma = \frac{(1-\sigma^2)(qJ\sigma + \mu B)\, d\beta}{1 - (1-\sigma^2)\beta q J}$$

$$\frac{1}{N} dE = -d\beta \frac{\{qJ\sigma + \mu B\}(qJ\sigma + \mu B)}{1 - (1-\sigma^2)\beta q J}$$

$$\frac{1}{N}\frac{dE}{dT} = \frac{1}{T}\frac{\left[\sigma T_c + \mu B\right]^2}{T - (1-\sigma^2)T_c}$$

5. Consider the one-dimensional Ising model. For the following, give analytic answers in terms of T, μB and J.

(a) What are the high B and low B limits for $\langle\sigma\rangle$?

(b) What are the high T and low T limits for $\langle\sigma\rangle$?

(c) Find an exact expression for the specific heat (per spin) in terms of T, μB and J.

(d) What are the high B and low $B = 0$ limits for the specific heat?

(e) What are the high T and low T limits for the specific heat?

$$\sigma = \frac{\sinh(\beta\mu B)}{\sqrt{\sinh^2(\beta\mu B) + e^{-4\beta J}}}$$

a) $\sigma(B\to\infty) = 1$

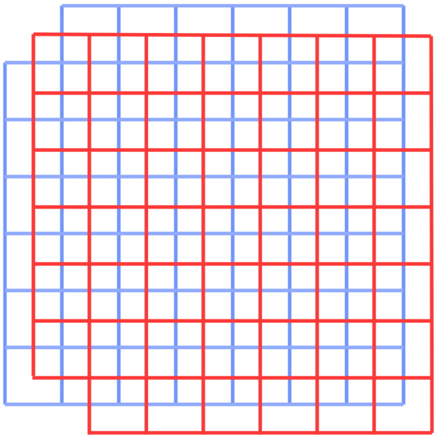
$$\sigma(B\to 0) = \beta\mu B\, e^{2\beta J}$$

b) $$\frac{1}{N}\ln Z = \ln\Big\{\underbrace{e^{\beta J}\cosh(\beta\mu B) + \sqrt{e^{2\beta J}\sinh^2(\beta\mu B) + e^{-2\beta J}}}_{\lambda}\Big\}$$

$$C_V = -\frac{1}{T^2}\frac{\partial(E/N)}{\partial\beta}$$
$$= \left(\frac{1}{\lambda}\frac{\partial^2\lambda}{\partial\beta^2} - \frac{1}{\lambda^2}\left(\frac{\partial}{\partial\beta}\lambda\right)^2\right)\frac{1}{T^2}$$

$$\frac{\partial\lambda}{\partial\beta} = J\cosh(\beta\mu B)e^{\beta J} + \mu B e^{\beta J}\sinh(\beta\mu B) + \frac{J\sinh^2(\beta\mu B)e^{2\beta J} + e^{2\beta J}\mu B\cosh(\beta\mu B)\sinh\beta\mu B - Je^{-2\beta J}}{\sqrt{\quad\cdots}}$$

$$\frac{\partial^2\lambda}{\partial\beta^2} = \left(J^2 + (\mu B)^2\right)\cosh(\beta\mu B)e^{\beta J} + 2\mu BJ\sinh(\beta\mu B)e^{\beta J} - 3\frac{\left[J\sinh^2\beta\mu B\, e^{2\beta J} + \mu B\cosh(\beta\mu B)\sinh\beta\mu B - Je^{-2\beta J}\right]^2}{\left[e^{2\beta J}\sinh^2\beta\mu B + e^{-2\beta J}\right]^{3/2}}$$
$$+ \frac{4\mu BJ\sinh(\beta\mu B)\cosh(\beta\mu B)e^{2\beta J} + e^{2\beta J}\left(2J^2\sinh^2(\beta\mu J) + (\mu B)^2\cosh(2\beta\mu J)\right) + 2J^2e^{-2\beta J}}{\left[e^{2\beta J}\sinh^2(\beta\mu B) + e^{-2\beta J}\right]^{1/2}}$$



6. Consider two-dimensional bond percolation of a large $L \times L$ simple square lattice. First consider the red lattice, where we will remove a percentage p of the red bonds. Next, we will remove all blue bonds that intersect a surviving red bond. Thus, in the limit of a large lattice, the fraction of blue bonds broken will be $(1 - p)$.

   (a) List which combinations of the following are possible.

      a) A connected string of blue bonds extends all the way from the bottom blue row to the top of the lattice.

      b) There is no connected string of blue bonds that extends all the way from the bottom to the top.

      c) A connected string of red bonds extends all the way from the left side to the right side of the red lattice.

      d) There is no connected string of red bonds that extends all the way from the left side to the right side of the red lattice.

   (b) What is p_c for a simple square lattice in bond percolation?

a) a, d & b, c

b) 0.5

7. Consider a one-dimensional bond percolation model of an infinitely long string, where the probability of any set of neighbors being connected is p.

   (a) In terms of p, find the probability that a fragment will have size A.

   (b) What is the average size of a fragment?

a) $P(A) = p^{A-1}(1-p)$

b) $$\langle A \rangle = \sum_A p^{(A-1)} A (1-p)$$
$$= (1-p)\frac{d}{dp}\sum_{A=1}^{\infty} p^A$$
$$= (1-p)\frac{d}{dp}\left[\frac{1}{1-p} - 1\right]$$
$$= \frac{1}{1-p}$$

check

$$\sum_{A=1}^{\infty} P(A) = (1-p)\sum_{A=1}^{\infty} p^{A-1}$$
$$= (1-p)\sum_{n=0}^{\infty} p^n = (1-p)\frac{1}{1-p} = 1 \checkmark$$

6 Chapter 6

6.1 Consider the example for which the surface energy was calculated, where

$$\Delta\Psi \equiv P_0 - P + (\mu - \mu_0)\rho = \frac{A}{2}[(\rho - \rho_c)^2 - \alpha^2]^2.$$

Using E. (6.17), solve for the density profile $\rho(x)$ between the two phases.

$$\frac{d}{d(\partial_x \rho)}\left[\frac{A[(\rho-\rho_c)^2-\alpha^2]^2}{2(\partial_x\rho)} - \frac{\kappa}{2}\partial_x\rho\right] = 0$$

let $\rho - \rho_c = \xi$, such that $\partial_x \xi = \partial_x \rho = \xi'$ and the previous equation can be re-written

$$\frac{d}{d\xi'}\left[\frac{A[\xi^2-\alpha^2]^2}{2\xi'} - \frac{\kappa}{2}\xi'\right] = 0$$

t

$$\left[\frac{A[\xi^2-\alpha^2]}{\xi'}(2\xi)\frac{d\xi}{d\xi'} - \frac{A[\xi^2-\alpha^2]^2}{2\xi'^2} - \frac{\kappa}{2}\right] = 0$$

$$\frac{d\xi'}{d\xi} = \frac{2\xi A}{\kappa}\left[\xi^2-\alpha^2\right](\xi')^{-1}$$

Then,

$$\begin{aligned}
\xi'^2 &= \frac{A[\xi^2-\alpha^2]^2}{\kappa} \\
\xi' &= \sqrt{\frac{A}{\kappa}}[\xi^2-\alpha^2] \\
\xi &= \frac{\tanh^{-1}(x/\alpha)}{\alpha} \\
\rho &= \frac{\tanh^{-1}(x/\alpha)}{\alpha} + \rho_c
\end{aligned} \tag{1}$$

6.2 Consider the one-dimensional Ising model, with the total energy in the mean field approximate being,

$$E = -\sum_i \frac{1}{2} qJ\langle\sigma\rangle\sigma_i.$$

(a) Let p = probability of $\sigma = +1$ and q = 1 - p = probability of $\sigma = -1$.
Then $\langle\sigma\rangle = p(+1) + q(-1) = 2p - 1$ and

$$\begin{aligned}
S/N &= -p\ln p - q\ln q \\
&= -\left[\frac{1+\langle\sigma\rangle}{2}\ln\left(\frac{1+\langle\sigma\rangle}{2}\right)\right] - \left[\frac{1-\langle\sigma\rangle}{2}\ln\left(\frac{1-\langle\sigma\rangle}{2}\right)\right]
\end{aligned} \tag{2}$$

(b)

$$\begin{aligned}\frac{d(F/N)}{d\langle\sigma\rangle} &= -qJ\langle\sigma\rangle + \frac{T}{2}\ln\left(\frac{1+\langle\sigma\rangle}{2}\right) - \frac{T}{2}\ln\left(\frac{1-\langle\sigma\rangle}{2}\right) \\ &= -qJ\langle\sigma\rangle + \frac{T}{2}\ln\left(\frac{1+\langle\sigma\rangle}{1-\langle\sigma\rangle}\right) \\ &= 0\end{aligned}$$

$$2\beta qJ\langle\sigma\rangle = \ln\left(\frac{1+\langle\sigma\rangle}{1-\langle\sigma\rangle}\right) \tag{3}$$

(c) This expression is equivalent to the previous expression:

$$\begin{aligned}(1-\langle\sigma\rangle) &= (1-\langle\sigma\rangle)e^{2\beta qJ\langle\sigma\rangle} \\ \langle\sigma\rangle &= \frac{e^{2\beta qJ\langle\sigma\rangle}-1}{e^{2\beta qJ\langle\sigma\rangle}+1} \\ &= \frac{e^{\beta qJ\langle\sigma\rangle}-e^{-\beta qJ\langle\sigma\rangle}}{e^{\beta qJ\langle\sigma\rangle}+e^{-\beta qJ\langle\sigma\rangle}} \\ &= \tanh(\beta qJ\langle\sigma\rangle)\end{aligned} \tag{4}$$

(d)

$$F/A = \int \rho_0\left[\nu(\sigma,T) + \frac{\kappa}{2}(\nabla\sigma)^2\right]dx - \int \rho_0\left[\nu(\sigma_{x=\pm\infty},T) + \frac{\kappa}{2}(\nabla\sigma)^2_{x=\pm\infty}\right]dx$$

The last term in this is 0 (see figure 6.1), and we limit our region of interest to $\sigma = \pm\sigma_{eq}$ so

$$\begin{aligned}F/A &= \int_{-\sigma_{eq}}^{\sigma_{eq}} \rho_0\left[\frac{\nu(\sigma,T)-\nu(\sigma_{eq},T)}{\partial_x\sigma} + \frac{\kappa}{2}(\partial_x\sigma)\right]d\sigma \\ &= \int_{-\sigma_{eq}}^{\sigma_{eq}} \rho_0\left[\frac{\mathcal{R}}{\partial_x\sigma} + \frac{\kappa}{2}(\partial_x\sigma)\right]d\sigma\end{aligned} \tag{5}$$

(e) We have to minimise the integrand in (5) to find the surface energy. Using 6.17,

$$\partial_x\sigma = \sqrt{\frac{2\mathcal{R}}{\kappa}}$$

(5) becomes

$$F/A = \sqrt{\frac{\kappa}{2}}\rho_0\int_{-\sigma_{eq}}^{\sigma_{eq}} 2\sqrt{\mathcal{R}}d\sigma \tag{6}$$

At T = 0, $\nu = -\frac{1}{2}T_c\sigma^2$ and $\nu(\sigma = \pm\sigma_{eq}) = -\frac{1}{2}qJ(\pm 1)^2 = -\frac{1}{2}T_c$ Then (6):

$$\begin{aligned}F/A &= \sqrt{2\kappa}\rho_0\int_{-1}^{1}\left[\sqrt{(T_c/2)(1-\sigma_{eq}^2)}\right]d\sigma \\ &= \sqrt{\kappa}\rho_0\sin^{-1}(1) \\ &= \frac{\sqrt{\kappa}\rho_0\pi}{2}\end{aligned} \tag{7}$$

6.3 Again we assume a quadratic potential

$$\mathcal{V}(x) = \frac{A\,|\phi(x)|^2}{2}. \tag{8}$$

The one-dimensional Fourier transform and inverse transform are defined as follows:

$$\tilde{\phi}_k \equiv \frac{1}{\sqrt{L}} \int_{-\infty}^{\infty} dx\, e^{ikx} \phi(x) \tag{9}$$

$$\phi(x) = \frac{1}{\sqrt{L}} \sum_k e^{-ikx} \tilde{\phi}_k. \tag{10}$$

The free energy can then be expressed as

$$\begin{aligned} F &= \frac{1}{2} \int_{-\infty}^{\infty} dx \left[A\,|\phi(x)|^2 + \kappa \left| \frac{\partial \phi}{\partial x} \right|^2 \right] && (11) \\ &= \frac{1}{2} \sum_k \left(A + \kappa k^2\right) \left| \tilde{\phi}_k \right|^2 && (12) \\ &= \frac{1}{2} \sum_k \left(A + \kappa k^2\right) \left[\left(\Re\mathrm{e}\, \tilde{\phi}_k\right)^2 + \left(\Im\mathrm{m}\, \tilde{\phi}_k\right)^2 \right]. && (13) \end{aligned}$$

The equipartition theorem then tells us that

$$\left\langle \frac{\left(A + \kappa k^2\right) \left(\Re\mathrm{e}\, \tilde{\phi}_k\right)^2}{2} \right\rangle = \left\langle \frac{\left(A + \kappa k^2\right) \left(\Im\mathrm{m}\, \tilde{\phi}_k\right)^2}{2} \right\rangle = \frac{T}{2}, \tag{14}$$

and so

$$\left\langle \left| \tilde{\phi}_k \right|^2 \right\rangle = \frac{2T}{A + \kappa k^2}. \tag{15}$$

Since distinct Fourier components are uncorrelated, we find the correlation of two Fourier components to be

$$\begin{aligned} \left\langle \tilde{\phi}_k^* \tilde{\phi}_{k'} \right\rangle &= \delta_{kk'} \left\langle \tilde{\phi}_k^* \tilde{\phi}_k \right\rangle && (16) \\ &= \delta_{kk'} \left\langle \left| \tilde{\phi}_k \right|^2 \right\rangle && (17) \\ &= \frac{2T\delta_{kk'}}{A + \kappa k^2}. && (18) \end{aligned}$$

Invoking (10) and (18) and replacing a sum with an integral, we have

$$\left\langle \phi\left(x\right)^{*}\phi\left(x'\right)\right\rangle = \frac{1}{L}\sum_{k}\sum_{k'}e^{i(kx-k'x')}\left\langle \tilde{\phi}_{k}^{*}\tilde{\phi}_{k'}\right\rangle \tag{19}$$

$$= \frac{1}{L}\sum_{k}\sum_{k'}e^{i(kx-k'x')}\frac{2T\delta_{kk'}}{A+\kappa k^{2}} \tag{20}$$

$$= \frac{2T}{L}\sum_{k}\frac{e^{ik(x-x')}}{A+\kappa k^{2}} \tag{21}$$

$$= \frac{2T}{L}\int_{-\infty}^{\infty}\frac{L\,dk}{2\pi}\frac{e^{ik(x-x')}}{A+\kappa k^{2}}. \tag{22}$$

Without loss of generality, we set $x'=0$:

$$\left\langle \phi\left(x\right)^{*}\phi\left(0\right)\right\rangle = \frac{T}{\pi}\int_{-\infty}^{\infty}\frac{dk\,e^{ikx}}{A+\kappa k^{2}}. \tag{23}$$

We can replace this integral with a contour integral in which k traverses counterclockwise a semicircle in the upper half of the complex plane, taking the radius of the semicircle to ∞ and noting that the extra piece of the contour we are adding does not contribute to the integral because the integrand vanishes as k approaches ∞ in any direction.

$$\left\langle \phi\left(x\right)^{*}\phi\left(0\right)\right\rangle = \frac{T}{\pi\kappa}\oint_{\Gamma}\frac{dk\,e^{ikx}}{\left(k+i\sqrt{\frac{A}{\kappa}}\right)\left(k-i\sqrt{\frac{A}{\kappa}}\right)} \tag{24}$$

where Γ is the contour of integration. Γ encloses a single singularity at $k=i\sqrt{\frac{A}{\kappa}}$. Thus by the residue theorem, we have

$$\left\langle \phi\left(x\right)^{*}\phi\left(0\right)\right\rangle = \frac{T}{\pi\kappa}\frac{2\pi i\exp\left(-\sqrt{\frac{A}{\kappa}}x\right)}{i\sqrt{\frac{A}{\kappa}}+i\sqrt{\frac{A}{\kappa}}} \tag{25}$$

$$= \frac{T}{\sqrt{\kappa A}}\exp\left(-\frac{x}{\xi}\right) \tag{26}$$

where $\xi\equiv\sqrt{\frac{\kappa}{A}}$ is the correlation length.

6.4 We are given the definition of the average density:

$$\left\langle \rho\right\rangle = \frac{\mathrm{Tr}\left[e^{-\beta H}e^{\beta\mu\int d^{3}r\,\rho(\mathbf{r})}\rho\left(\mathbf{0}\right)\right]}{\mathrm{Tr}\left[e^{-\beta H}e^{\beta\mu\int d^{3}r\,\rho(\mathbf{r})}\right]}. \tag{27}$$

The derivative of $\langle \rho \rangle$ with respect to μ can be evaluated via the derivative quotient rule:

$$
\begin{aligned}
\frac{d\langle\rho\rangle}{d\mu} = & \left\{\frac{d}{d\mu}\operatorname{Tr}\left[e^{-\beta H}e^{\beta\mu\int d^3r\,\rho(\mathbf{r})}\rho\left(\mathbf{0}\right)\right]\operatorname{Tr}\left[e^{-\beta H}e^{\beta\mu\int d^3r\,\rho(\mathbf{r})}\right]\right. \\
& \left. -\operatorname{Tr}\left[e^{-\beta H}e^{\beta\mu\int d^3r\,\rho(\mathbf{r})}\rho\left(\mathbf{0}\right)\right]\frac{d}{d\mu}\operatorname{Tr}\left[e^{-\beta H}e^{\beta\mu\int d^3r\,\rho(\mathbf{r})}\right]\right\} \\
& \cdot\left\{\operatorname{Tr}\left[e^{-\beta H}e^{\beta\mu\int d^3r\,\rho(\mathbf{r})}\right]\right\}^{-2}
\end{aligned} \tag{28}
$$

$$
\begin{aligned}
= & \left\{\operatorname{Tr}\frac{d}{d\mu}\left[e^{-\beta H}e^{\beta\mu\int d^3r\,\rho(\mathbf{r})}\rho\left(\mathbf{0}\right)\right]\operatorname{Tr}\left[e^{-\beta H}e^{\beta\mu\int d^3r\,\rho(\mathbf{r})}\right]\right. \\
& \left. -\operatorname{Tr}\left[e^{-\beta H}e^{\beta\mu\int d^3r\,\rho(\mathbf{r})}\rho\left(\mathbf{0}\right)\right]\operatorname{Tr}\frac{d}{d\mu}\left[e^{-\beta H}e^{\beta\mu\int d^3r\,\rho(\mathbf{r})}\right]\right\} \\
& \cdot\left\{\operatorname{Tr}\left[e^{-\beta H}e^{\beta\mu\int d^3r\,\rho(\mathbf{r})}\right]\right\}^{-2}.
\end{aligned} \tag{29}
$$

(The trace and derivative commute.)

$$
\begin{aligned}
\frac{d\langle\rho\rangle}{d\mu} = & \left(\operatorname{Tr}\left\{e^{-\beta H}\beta\left[\int d^3r\,\rho\left(\mathbf{r}\right)\right]e^{\beta\mu\int d^3r\,\rho(\mathbf{r})}\rho\left(\mathbf{0}\right)\right\}\right. \\
& \cdot\operatorname{Tr}\left[e^{-\beta H}e^{\beta\mu\int d^3r\,\rho(\mathbf{r})}\right]-\operatorname{Tr}\left[e^{-\beta H}e^{\beta\mu\int d^3r\,\rho(\mathbf{r})}\rho\left(\mathbf{0}\right)\right] \\
& \left.\cdot\operatorname{Tr}\left\{e^{-\beta H}\beta\left[\int d^3r\,\rho\left(\mathbf{r}\right)\right]e^{\beta\mu\int d^3r\,\rho(\mathbf{r})}\right\}\right) \\
& \cdot\left\{\operatorname{Tr}\left[e^{-\beta H}e^{\beta\mu\int d^3r\,\rho(\mathbf{r})}\right]\right\}^{-2}
\end{aligned} \tag{30}
$$

$$
\begin{aligned}
= & \left(\operatorname{Tr}\left\{e^{-\beta H}\beta\left[\int d^3r\,\rho\left(\mathbf{r}\right)\right]e^{\beta\mu\int d^3r\,\rho(\mathbf{r})}\rho\left(\mathbf{0}\right)\right\}\right. \\
& \cdot\operatorname{Tr}\left[e^{-\beta H}e^{\beta\mu\int d^3r\,\rho(\mathbf{r})}\right]-\operatorname{Tr}\left[e^{-\beta H}e^{\beta\mu\int d^3r\,\rho(\mathbf{r})}\rho\left(\mathbf{0}\right)\right] \\
& \left.\cdot\operatorname{Tr}\left\{e^{-\beta H}\beta\left[\int d^3r\,\rho\left(\mathbf{r}\right)\right]e^{\beta\mu\int d^3r\,\rho(\mathbf{r})}\right\}\right) \\
& \cdot\left\{\operatorname{Tr}\left[e^{-\beta H}e^{\beta\mu\int d^3r\,\rho(\mathbf{r})}\right]\right\}^{-2}
\end{aligned} \tag{31}
$$

$$
\begin{aligned}
= & \frac{\beta\operatorname{Tr}\left[e^{-\beta H}e^{\beta\mu\int d^3r\,\rho(\mathbf{r})}\int d^3r\,\rho\left(\mathbf{r}\right)\rho\left(\mathbf{0}\right)\right]}{\operatorname{Tr}\left[e^{-\beta H}e^{\beta\mu\int d^3r\,\rho(\mathbf{r})}\right]} \\
& -\frac{\beta\operatorname{Tr}\left[e^{-\beta H}e^{\beta\mu\int d^3r\,\rho(\mathbf{r})}\rho\left(\mathbf{0}\right)\right]\operatorname{Tr}\left[e^{-\beta H}e^{\beta\mu\int d^3r\,\rho(\mathbf{r})}\int d^3r\,\rho\left(\mathbf{r}\right)\right]}{\left\{\operatorname{Tr}\left[e^{-\beta H}e^{\beta\mu\int d^3r\,\rho(\mathbf{r})}\right]\right\}^2}
\end{aligned} \tag{32}
$$

$$
= \beta\left\langle\rho\left(\mathbf{r}\right)\rho\left(\mathbf{0}\right)\right\rangle-\beta\left\langle\rho\left(\mathbf{r}\right)\right\rangle\left\langle\rho\left(\mathbf{r}\right)\right\rangle. \tag{33}
$$

In this last step, I am assuming that the expected value of $\rho\left(\mathbf{r}\right)$ is the same for all points and thus equals the expected value of $\rho\left(\mathbf{0}\right)$. Next, multiplying both sides by $T=\beta^{-1}$, we arrive at

$$
T\frac{d\langle\rho\rangle}{d\mu}=\left\langle\rho\left(\mathbf{r}\right)\rho\left(\mathbf{0}\right)\right\rangle-\left\langle\rho\left(\mathbf{r}\right)\right\rangle^2. \tag{34}
$$