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PHY 321, Fall 2023

Lagrangian - Small Vibration

1. Consider a block of mass **m** attached to a spring with a spring constant of **k**. The mass is displaced from its equilibrium by a distance $\mathbf{x_0}$, and released with an initial velocity of zero.
    a. Write down the kinetic and potential energies of the system.
    b. Write down the Lagrangian for the system.
    c. Write the equation of motion for this system.
    d. Solve the differential equation of motion to get the position of the mass on the spring as a function of time.

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Solutions

1.

a. $KE = \frac{1}{2}mv_x^2 = \frac{1}{2}m\dot{x}^2$ $\qquad U = \frac{1}{2}kx^2$

b. Due to the fact that the system has no damping and friction, we can make a Lagrangian for the system and apply the Euler-Lagrange equation.

$L = KE - U$

$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$

c. $\frac{d}{dt}\left(\frac{dL}{d\dot{x}}\right) = \frac{dL}{dx}$

$$\frac{dL}{dx} = 0 - kx = -kx$$

$$\frac{dL}{d\dot{x}} = m\dot{x} - 0 = m\dot{x}$$

$$\frac{d}{dt}(m\dot{x}) = m\ddot{x}$$

After rearranging this, we get: $m\ddot{x} = -kx$

$$m\ddot{x} + kx = 0$$

d. This equation is of the form $x(t) = Acos(\omega_0 t - \varphi)$, where A is the amplitude of the spring's oscillation, φ is the phase constant, and ω_0 is equal to $\sqrt{\frac{k}{m}}$

The spring is stretched to a displacement of x_0 and then it is released from there, so A = x_0 . The phase constant φ is zero because the mass is released from rest and it has its largest amount of displacement in the positive x direction at t = 0. Therefore, the position as a function of time is:

$$x(t) = x_0 \cos(\omega_0 t)$$