Nick Majeski
Curtis Chou

PHY 321 Group Review Problem:

Consider a star in a binary orbit with a stellar-mass black hole. The star has a density ρ_s and radius r_s. The black hole has mass M_{bh} and radius r_{sch}. The distance from the center of the star to the black hole singularity is D. A pocket of plasma with mass δm sits on the surface of the star.

(a) If the pocket of plasma sits at a point facing either away or directly towards the black hole, express the tidal force on the plasma pocket in terms of G, M_{bh}, r_{sch}, r_s, and δm.

(b) Use the gravitational force and the tidal force to derive an expression for the Roche limit D_ℓ, the distance at which the black hole rips apart the star in terms of M_{bh}, D, and ρ_s.

(c) If $\rho_s = \rho_\odot \approx 1410 \frac{kg}{m^3}$ what is the minimum mass the black hole must have for the star to cross the event horizon without being torn apart (spaghettified)? Note the Schwarzschild radius is given by $r_{schw} = \frac{2GM_{bh}}{c^2}$, where c is the speed of light in a vacuum.

(d) By expressing the answer for (b) in terms of r_{schw}, ρ_s and ρ_{bh}, the density of the black hole, if $\rho_s = \rho_{bh}$, does the star get torn apart before crossing the event horizon?

$$\vec{F} = m\frac{d^2 r'}{dt^2}$$

a) express the tidal force of the pocket of plasma in terms of G, M_{bh}, R_{sch}, r_s, and, δm

$$\vec{F}_{eff} \approx 2\frac{GM\delta m}{D^3} r\cos\theta\,\hat{z} + \text{other forces acting on } \delta m$$

$\cos\theta = 1$ if pointing in $\hat{z}$.

and $\cos\theta = -1$ " in $-\hat{z}$

$$F_{tidal} = \frac{2GM_{bh}\delta m}{D^3} r$$

Derivation

Inertial refrence frame:

$$m\frac{d^2\vec{r}'}{dt^2} = \vec{F}$$

(ma)

If we set a second coordinate system

$\vec{r} = \vec{r}' - \vec{r}_0$, $\quad \vec{r}_0 = \frac{1}{2}\vec{a}_0 t^2$.

$$\therefore m\frac{d^2\vec{r}}{dt^2} = \vec{F} - m\vec{a}_0$$

$\vec{r}_0 = \frac{1}{2}\vec{a}_0 t^2$ is

derived from

$s = ut + \frac{1}{2}at^2$. (distance under constant acceleration)

$$\therefore m\frac{d^2\vec{r}}{dt^2} = \vec{F} - m\vec{a}_0$$

b) De Roache limit in terms of
M_{bh}, D, and ρ_s.

star (density of star)
singularity

If we preform taylor expansion similar to

$$F_g = \frac{G V_s \rho_s \delta m}{r_s^2} = \frac{G(\frac{4}{3}\pi r_s^3)\rho_s \delta m}{r_s^2}$$

$$= \frac{4\pi G r_s \rho_s \delta m}{3}$$

* r_s - star radius
V_s - star volume
ρ_s - star density
δm - Mass of plasmid packet

for

$$F_g = \frac{G m_1 m_2}{r^2}$$

$$F_{tidal} = F_g \Rightarrow \frac{2 G M_{bh} \delta m}{D_e^3} r_s = \frac{4\pi G r_s \rho_s \delta m}{3}$$

$$\Rightarrow \frac{2 M_{bh}}{D_e^3} = \frac{4\pi \rho_s}{3}$$

$$\therefore D_e = \sqrt[3]{\frac{3 M_{bh}}{2\pi \rho_s}}$$

CamScanner

Solution:

(a) $F_{tidal} = \frac{2GM_{bh}\delta m}{D^3} r_s$

(b) $F_g = \frac{GV_s\rho_s\delta m}{r_s^2} = \frac{G(\frac{4}{3}\pi r_s^3)\rho_s\delta m}{r_s^2} = \frac{4\pi G r_s\rho_s\delta m}{3}$

$F_{tidal} = F_g \implies \frac{2GM_{bh}\delta m}{D_\ell^3} r_s = \frac{4\pi G r_s \rho_s \delta m}{3}$

$\iff \frac{2M_{bh}}{D_\ell^3} = \frac{4\pi\rho_s}{3} \iff D_\ell = \sqrt[3]{\frac{3M_{bh}}{2\pi\rho_s}}$

(c) $D_\ell < r_{schw} \implies \sqrt[3]{\frac{3M_{bh}}{2\pi\rho_s}} < \frac{2GM_{bh}}{c^2}$

$\iff \frac{c^2}{2G}\sqrt[3]{\frac{3}{2\pi\rho_s}} < (M_{bh})^{2/3} \iff M_{bh} > \sqrt{\frac{3c^6}{16G^3\pi\rho_s}}$

$M_{bh} > \sqrt{\frac{3(3\times10^8\ m/s)^6}{16(6.67\times10^{-11}\frac{m^3}{kg\cdot s^2})^3\pi(1410\frac{kg}{m^3})}} = 3.22\times10^{38}\ kg = 1.620\times10^8 M_\odot$

(d) $D_\ell = \sqrt[3]{\frac{3(\frac{4}{3}\pi r_{schw}^3)\rho_{bh}}{2\pi\rho_s}} = \sqrt[3]{\frac{2\rho_{bh}}{\rho_s}}\cdot r_{schw}$

If $\rho_{bh} = \rho_s$, then $D_\ell = \sqrt[3]{2}\cdot r_{schw} > r_{schw}$. Thus, the star would be torn apart by the black hole before crossing the event horizon.