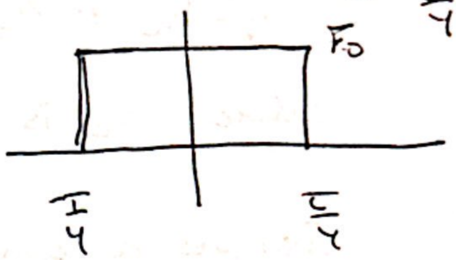
Consider a mass m attached to a spring with constant k with minimal damping. Consider also a periodic force applied so that the equation of motion is:

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = F(t)/m \qquad F(t) = \begin{cases} 0 & -\frac{\tau}{2} < t < -\frac{\tau}{4} \\ F_0 & -\frac{\tau}{4} < t < \frac{\tau}{4} \\ 0 & \frac{\tau}{4} < t < \frac{\tau}{2} \end{cases}$$

(a) Find Fourier coefficients eg. f_n, g_n

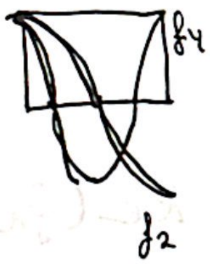


$$F(t) = \frac{f_0}{2} + \sum f_n \cos n\omega t + g_n \sin(n\omega t)$$

$\omega = \frac{2\pi}{\tau}$

All g_n's are ~~odd~~ zero since function is even.

All even f_n are zero since even f_n terms spend equal time above and below zero.



$$f_0 = F_0 \quad \text{since} \quad \frac{f_0}{2} = \langle F(t) \rangle = \frac{F_0}{2}$$

$$f_n = \frac{4F_0}{\tau}\int_0^{\frac{\tau}{4}} \cos n\omega t \, dt = \frac{2F_0}{n\pi}\sin\left(n\frac{\pi}{2}\right)$$

(b) Solve for $x(t)$ at large times (eg. $x_p(t)$)

$$L \equiv \left(\frac{d^2}{dt^2} + K\frac{d}{dt} + \omega_0^2\right) \quad \text{so} \quad Lx = F(t)/m = \frac{1}{m}\left(\sum f_n \cos n\omega t + \frac{f_0}{2}\right)$$

Since L linear, if x_n satisfies

$$Lx_n = \frac{f_n \cos n\omega t}{m}$$

Then $L\sum_n x_n = \sum_n Lx_n = \sum \frac{f_n}{m}\cos n\omega t$

So $\sum_n x_n + x_0$ solves $Lx = F(t)/m$

where x_0 is solution to $Lx = \frac{f_0/m}{2}$

x_0: Just guess a constant D_0

$$L(D_0) = \frac{f_0/m}{2} = \omega_0^2 D_0$$

$$D_0 = \frac{f_0/m}{2\omega_0^2}$$

x_n: Note ~~$\sum f_n \cos n$~~

$$f_n \cos n\omega t = \mathrm{Re}(f_n e^{in\omega t})$$

Since L is linear and real, if we solve

$Lx_n' = f_n e^{in\omega t}$, then the solution

$x_n = \mathrm{Re}(x_n')$.

~~$So\ Lx_n' = f_n e^{in\omega t}$~~. So guess $C_n e^{in\omega t} = x_n$

$$Lx_n = C_n \cancel{e^{in\omega t}}\left(-(n\omega)^2 + \omega_0^2 + 2\beta i n\omega\right) = \frac{f_n}{m}\cancel{e^{in\omega t}}$$

$$C_n = \frac{f_n/m}{\omega_0^2 - (\omega n)^2 + 2\beta i n\omega}$$

$$\text{So } x_p(t) = D_0 + \mathrm{Re}\left(\sum \frac{f_n/m}{\omega_0^2 - (n\omega)^2 + 2\beta i n\omega}\right)$$