Question:

A ball sits stable on top of a hill with potential $U(x) = kx^2 - kx^4/a^2$, where k and alpha are integers, k is positive, and x is the displacement of the ball. Find the force on the ball, and describe its behavior.

Solution:

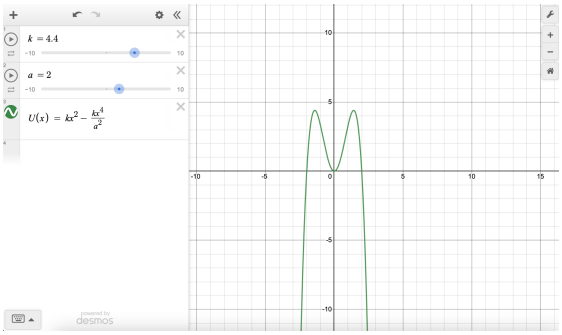
This is an example of an object that could be in unstable equilibrium. In stable equilibrium, an object will fall back to its original potential when it's displaced by a force, like a ball rolling in between two hills. In unstable equilibrium, an object will continue to move away from its original potential when it's displaced.

To find the force of the ball in disequilibrium, take the derivative of $U(x)$ with respect to x.

$$U(x) = kx^2 - kx^4/a^2$$
$$-(d/dx)(U(x)) = -2kx + 4kx^3/a^2 = F(x)$$

Next, find the extrema of $U(x)$ by setting $F(x) = 0$.

$$-2kx + 4kx^3/a^2 = 0$$
$$-2x(1 - 2x^2/a^2) = 0$$
$$1 = 2x^2/a^2$$
$$0.5a^2 = x^2$$
$$x = 0, +/- 0.71a$$



At position 0, the ball is in stable equilibrium since it's at the potential minimum, but at position $+/- 0.71a$ (the potential maxima) the ball is in unstable equilibrium. If the ball experiences a force in the positive x direction at $0.71a$, the ball will fall and theoretically reach infinite negative potential (with real constraints), and conversely at $-0.71a$. If the ball experiences a leftwards force at at at position $0.71a$, less than what's required for it to reach the next maximum at $-0.71a$, it will fall into stable equilibrium eventually at position 0.