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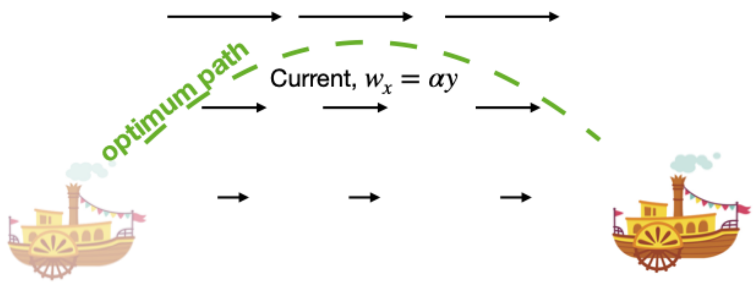
Exercise, Monday, Nov. 6, 2023



You wish to find the fastest route for a paddle boat, which moves with speed V_0 on still water, to travel a distance L, from $x = y = 0$ to $x = L, y = 0$. There is a current also moving in the x direction, but the current's velocity depends on y. The current's speed is $w_x = \alpha y$. Rather than minimizing the time to go a distance L, we will maximize the distance L one travels in a net time T. This will result in the same trajectory once the time is adjusted so the distance is indeed L. That distance is

$$L = \int_0^t dt' \frac{dx}{dt}(y, y'), \quad y' \equiv dy/dt$$

Our goal is use the Euler-Lagrange equation to solve for $y(t)$.

1. (10 pts) Express dx/dt in terms of w_x, V_0 and y'.

2. (10 pts) Using the expression for x above, write a 2nd-order differential equation for $y(t)$ from the Euler-Lagrange equations. The equation should be in terms of L, α and w_x, y, d^2y/dt^2 and dy/dt. Arrange answer in the form:

$$\frac{dv_y}{dt} = \text{function of everything else including } v_y$$

3. (5 pts) Using symmetry arguments, at what time does $v_y = 0$?

4. (10 pts) The solution to the above differential equation is:

$$v_y(t') = -V_0 \frac{\alpha t'}{\sqrt{1 + \alpha^2 t'^2}},$$
$$t' = t - T/2.$$

Edit the python template and plot x vs t. Use $T = 1$ hour, $V_0 = 10$ m/s, and $\alpha = 0.001$ s^{-1}.

If you were to solve everything analytically:
To solve this first write equation as

$$\frac{dy'}{(V_0^2 - y'^2)^{3/2}} = -\alpha V_0^2 dt,$$

Integrate both sides. But to make life easier, work in terms of variable $t' = t - T/2$, because by symmetry $y'(t' = 0) = y'(t = T/2) = 0$. Then

$$\int_0^{y'/V_0} \frac{du}{(1-u^2)^{3/2}} = -\alpha t'.$$

To solve numerically, this give t' for a given y'. One can then integrate y' to get $y(t')$. Because $y(t' = \pm T/2) = 0$, one subtracts $y'(t' = T/2)$ from the result for $y'(t')$ to get $y(t')$. Once one knows y, along with y' then one can express $v_x(t')$. Finally, one can integrate $v_x(t')$ to get $x'(t')$, then subtract $x'(-T/2)$ to get $x(t')$. This is not an easy program to write.
Alternatively, one can also solve for all the relations analytically.

$$\begin{aligned}
\frac{u}{\sqrt{1-u^2}} &= -\alpha t', \quad u = y'/V_0 \\
y'(t') &= -V_0 \frac{\alpha t'}{\sqrt{1+\alpha^2 t'^2}}, \\
t' &= t - T/2.
\end{aligned}$$

Next, integrate $y'(t')$ to get $y(t')$. Choose arbitrary constant to ensure that $y(t' = -T/2) = y(t' = T/2) = 0$.

$$\begin{aligned}
y(t') &= \frac{-V_0}{\alpha}\sqrt{1+\alpha^2 t'^2} + \frac{V_0}{\alpha}\sqrt{1+\alpha^2 (T/2)^2}, \\
t' &= t - T/2.
\end{aligned}$$

Next, write down v_x,

$$\begin{aligned}
v_x(t') &= \alpha y + V_0\sqrt{1 - \frac{\alpha^2 t'^2}{1+\alpha^2 t'^2}}, \\
&= V_0\sqrt{1+\alpha^2(T/2)^2} + V_0\left\{-\sqrt{1+\alpha^2 t'^2} + \sqrt{1-\alpha^2 t'^2/(1+\alpha^2 t'^2)}\right\} \\
&= V_0\sqrt{1+\alpha^2(T/2)^2} + V_0\left\{-\sqrt{1+\alpha^2 t'^2} + 1/\sqrt{1+\alpha^2 t'^2}\right\}, \\
t' &= t - T/2.
\end{aligned}$$

Finally, integrate v_x to get $x(t')$. But first do an integral

$$\begin{aligned}
\int du\ \sqrt{1+u^2} &= \int d\theta \cosh^2\theta, \quad u = \sinh\theta, \\
&= \frac{1}{2}\int d\theta(1+\cosh 2\theta) = \frac{1}{2}\left[\theta + \sinh 2\theta/2\right] \\
\int du\ \sqrt{1+u^2} &= \frac{1}{2\alpha}\left[\sinh^{-1}(\alpha t') + \alpha t'\sqrt{1+\alpha^2 t'^2}\right].
\end{aligned}$$

The other integral yields an inverse sinh, and

$$\tilde{x}(t') = V_0 t'\sqrt{1+\alpha^2(T/2)^2} + \frac{V_0}{2\alpha}\sinh^{-1}(\alpha t') - \frac{V_0 t'}{2}\sqrt{1+\alpha^2 t'^2}$$

The quantity $\tilde{x}$ is the distance from the position at $T/2$. To get x, one must add back in $x(t' = T/2)$. Here, $x(t = T) = 2x(t' = T/2)$,

$$\begin{aligned}
x &= \tilde{x}(t') + x(t = T)/2, \\
t' &= t - T/2, \\
x(t = T) &= \frac{V_0 T}{2}\sqrt{1+\alpha^2(T/2)^2} + \frac{V_0}{\alpha}\sinh^{-1}(\alpha T/2).
\end{aligned}$$