your name____

Physics 321 Midterm #2 - Friday, Nov. 10, 2023

Some integrals:

$$\begin{aligned}
\int \frac{dx}{1+x^2} &= \tan^{-1}(x),\\
\int \frac{dx}{1-x^2} &= \tanh^{-1}(x),\\
\int \frac{dx}{\sqrt{1-x^2}} &= \sin^{-1}(x),\\
\int \frac{dx}{\sqrt{1+x^2}} &= \sinh^{-1}(x),\\
\int dx\ \tan(x+\phi) &= -\ln(\cos(x+\phi))\\
\int dx\ \tanh(x+a) &= \ln(\cosh(x+a)).
\end{aligned}$$

$$\begin{aligned}
F(t) &= \frac{f_0}{2} + \sum_n f_n \cos(n\omega t) + g_n \sin(n\omega t),\\
f_n &= \frac{2}{\tau}\int_{-\tau/2}^{\tau/2} dt\ F(t)\cos(2n\pi t/\tau),\\
g_n &= \frac{2}{\tau}\int_{-\tau/2}^{\tau/2} dt\ F(t)\sin(2n\pi t/\tau).
\end{aligned}$$

For the equation

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0,$$

The solutions are:

$$\begin{aligned}
x &= Ae^{i\omega t},\\
\omega &= i\beta \pm \sqrt{\omega_0^2 - \beta^2},\\
x(t) &= Ae^{-\beta t}e^{\pm i\sqrt{\omega_0^2-\beta^2}t},\ \text{underdamped},\\
&= Ae^{-\beta t \pm \sqrt{\beta^2-\omega_0^2}t},\ \text{overdamped}.
\end{aligned}$$

Write your name on EVERY page!

80 points possible, plus 25 points extra credit

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your name______________________________________

1. A positively charged particle, with charge $\boldsymbol{q}$, is accelerated by passing through a voltage $\boldsymbol{V_K}$. The acceleration leaves the charged particle with a kinetic energy $\boldsymbol{KE = qV_K}$. The particle then approaches a small conducting sphere of radius $\boldsymbol{R}$, which is at a positive voltage $\boldsymbol{V_{\mathbf{sphere}}}$. The potential energy of the particle at the surface is thus $\boldsymbol{qV_{\mathbf{sphere}}}$ (voltage measured relative to a point far from the sphere, so the electric field from the sphere behaves as if it were caused by a point charge as long as you are outside the sphere.)

   (a) (5 pts) What is the minimum accelerating voltage, $\boldsymbol{V_{K,\mathbf{min}}}$, necessary for the particle to reach the sphere?

   (b) (15 pts) What is the cross section for colliding with the sphere? Assume $\boldsymbol{V_K > V_{K,\mathbf{min}}}$. Show your work. Derive your answer by beginning with energy and angular momentum conservation, and express your answer in terms of $\boldsymbol{R, V_K}$ and $\boldsymbol{V_{\mathbf{sphere}}}$.

your name________________________________

Extra Space for No.1

Solution: a) $V_{K,\min} = V_{\text{sphere}}$

b) Use conservation of angular momentum and energy for a trajectory that grazes the sphere. Label the impact parameter as $b_{\max}$. Label the incoming velocity as v_0 and the speed when it grazes the sphere as v_f.

$$
\begin{aligned}
mv_0 b_{\max} &= mv_f R_{\text{sphere}}, \\
\frac{1}{2}mv_0^2 &= \frac{1}{2}mv_f^2 + qV_{\text{sphere}}, \\
b_{\max} &= R\frac{v_f}{v_0} \\
v_0^2 &= \frac{2q}{m}V_K, \\
v_f^2 &= \frac{2q}{m}(V_K - V_{\text{sphere}}), \\
b_{\max} &= R\sqrt{\frac{(2q/m)V_K - (2q/m)V_{\text{sphere}}}{(2q/m)V_K}}, \\
&= R\sqrt{\frac{V_K - V_{\text{sphere}}}{V_K}}, \\
\sigma &= \pi b^2 = \pi R^2 \left(1 - \frac{V_{\text{sphere}}}{V_K}\right).
\end{aligned}
$$

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2. (20 pts, no points awarded if answer has wrong dimension) You are an experimentalist planning a scattering experiment using a gold target and a much lighter projectile. You go online and find the mass density of gold, $\boldsymbol{\rho_{\mathrm{Au}}}$, and the mass of a gold atom, $\boldsymbol{m_{\mathrm{Au}}}$. Somebody sends you a target, but they didn't tell you the dimensions of the foil, only that it is sufficiently thin you can treat the problem in the thin limit. You measure the width and length, $\boldsymbol{W}$ and $\boldsymbol{L}$, then measure the mass of the foil, $\boldsymbol{M}$, but you can't measure the thickness. The accelerator provides a beam, with $\boldsymbol{N_{\mathrm{beam}}}$ particles incident on the foil over the duration of the experiment. One of your detectors is positioned at a scattering angle $\boldsymbol{\theta_s = 45^\circ}$. The detector has a cross-sectional area $\boldsymbol{a}$ and is placed a distance $\boldsymbol{R}$ from the center of the target. Over the entire experiment you record $\boldsymbol{N_{\mathrm{counts}}}$ in the detector. What is $\boldsymbol{d\sigma/d\Omega}$ for $\boldsymbol{\theta_s = 45^\circ}$?

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Extra Space for No.2

Solution: Probability of scattering into detector element is

$$\mathbf{Prob} = \frac{N_{\text{target atoms}} d\sigma/d\Omega\delta\Omega}{A}.$$

The number of target atoms per area is

$$\frac{N_{\text{target atoms}}}{A} = \frac{M}{mWL}.$$

So, the Probability is

$$\mathbf{Prob} = \frac{M}{mWL}\frac{d\sigma}{d\Omega}\Delta\Omega.$$

The angular coverage of the detector is

$$\Delta\Omega = \frac{a}{R^2},$$

and the probability is $N_{\text{counts}}/N_{\text{beam}}$. This gives

$$\frac{d\sigma}{d\Omega} = \frac{N_{\text{counts}}}{N_{\text{beam}}}\frac{R^2}{a}\frac{mWL}{M}.$$

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3. A mass m is connected to a spring with spring constant $k = m\omega_0^2$ and has a very small damping constant. It is subject to an additional periodic force,

$$F(t) = \left\{ \begin{array}{cc} 0, & -\tau/2 < t < 0, \\ F_0, & 0 < t < \tau/2 \end{array} \right. ,$$
$$F(t+\tau) = F(t).$$

Expressing the force as a series

$$F(t) = \frac{f_0}{2} + \sum_n f_n \cos(n\omega t) + g_n \sin(n\omega t),$$
$$\omega = 2\pi/\tau,$$

(a) (10 pts) Which coefficients are zero?

(b) (10 pts) What are the non-zero coefficients f_n and g_n? Note that $\omega \neq \omega_0$.

(c) (10 pts) Assuming the force has been active for a very very long time (long enough that the very small damping eliminates any dependence on the initial conditions), express the displacement $x(t)$ in terms of the coefficients (as a sum). You can leave your answers in terms of f_n and g_n to avoid double penalty should your answers to the first two parts be incorrect.

your name____________________________

Extra Space for No.3

Solution: a) One can rewrite the force as

$$F(t) = \frac{F_0}{2} + F_{\rm odd}(t),$$

where the odd piece is

$$F_{\rm odd}(t) = \begin{cases} -F_0/2, & -\tau/2 < t < 0, \\ F_0/2, & 0 < t < \tau/2 \end{cases}.$$

b)

$$\begin{aligned} f_0 &= \frac{F_0}{2}, \\ f_{n\neq 0} &= 0 \ \text{ integrating odd function gives zero}, \\ g_n &= \frac{2}{\tau}2\int_0^{\tau/2} dt\ \sin(n\omega t)F_0/2, \\ &= \frac{2F_0}{n\omega\tau}[1-\cos(n\omega\tau/2)] \\ &= \frac{2F_0}{n\omega\tau}[1-\cos(n\pi)] \\ &= \begin{cases} \frac{2F_0}{\pi n} & n = 1,3,5\cdots \\ 0 & n = 2,4,6\cdots \end{cases} \end{aligned}$$

c) Consider a single term in the force, $g_n \sin(n\omega t)$. Assume the particular solution is of the form (remember damping is small)

$$x_n(t) = A_n \sin(n\omega t).$$

Putting that into the differential equation,

$$\begin{aligned} -(n^2\omega^2)A_n \sin(n\omega t) + \omega_0^2 A_n \sin(n\omega t) &= \frac{g_n}{m}\sin(n\omega t), \\ A_n &= \frac{g_n/m}{\omega_0^2 - n^2\omega^2}. \end{aligned}$$

$$\begin{aligned} x(t) &= \frac{F_0}{2m\omega_0^2} + \sum_{n=1,3,5\cdots} \frac{g_n/m}{(\omega_0^2 - n^2\omega^2)}\sin(n\omega t), \\ &= \frac{F_0}{2m\omega_0^2} + \sum_{n=1,3,5\cdots} \frac{2F_0}{\pi n m(\omega_0^2 - n^2\omega^2)}\sin(n\omega t). \end{aligned}$$

The first term is simply the displacement due to a constant force F_0 on a spring with $k = m\omega_0^2$.