your name______________________________________

Physics 321 Midterm #1 - Friday, Sep. 29, 2023

Some integrals:

$$\begin{aligned}
\int \frac{dx}{1+x^2} &= \tan^{-1}(x), \\
\int \frac{dx}{1-x^2} &= \tanh^{-1}(x), \\
\int \frac{dx}{\sqrt{1-x^2}} &= \sin^{-1}(x), \\
\int \frac{dx}{\sqrt{1+x^2}} &= \sinh^{-1}(x), \\
\int dx\ \tan(x+\phi) &= -\ln(\cos(x+\phi)) \\
\int dx\ \tanh(x+a) &= \ln(\cosh(x+a)).
\end{aligned}$$

Write your name on EVERY page!

80 points possible, plus 25 points extra credit

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your name______________________________________

1. A particle of mass $\boldsymbol{m}$ moves under the influence of a potential energy,

$$\boldsymbol{U(x) = \alpha|x|}.$$

The particle starts at the origin moving with velocity $\boldsymbol{v_0}$.

(a) (5 pts) What is the particle's speed as a function of its position, $\boldsymbol{x}$?

(b) (15 pts) How much time, $\boldsymbol{t}$, passes before the particle turns around?

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Solution: a)
Use energy conservation,

$$\frac{1}{2}mv^2 + \alpha x = \frac{1}{2}mv_0^2,$$
$$v(x) = \sqrt{v_0^2 - 2\alpha x/m}.$$

b)

$$\begin{aligned}
t &= \int_0^{x_{\max}} \frac{dx}{v(x)} \\
&= \int_0^{x_{\max}} \frac{dx}{\sqrt{v_0^2 - 2\alpha x/m}}, \\
x_{\max} &= \frac{mv_0^2}{2\alpha}, \\
t &= -\sqrt{m/2\alpha}\int_0^{x_{\max}} \frac{dx}{\sqrt{x_{\max} - x}} \\
&= -\,2\sqrt{(m/2\alpha)}\sqrt{(x_{\max} - x)}\Big|_0^{x_{\max}} \\
&= 2\sqrt{\frac{mx_{\max}}{2\alpha}} \\
&= \frac{mv_0}{\alpha}.
\end{aligned}$$

There is a simpler way to do this if you recognize that the linear potential gives constant acceleration, $a = \alpha/m$. The time is then

$$t = \frac{v_0}{a} = \frac{mv_0}{\alpha}.$$

Note that this approach is less general and only works here because the potential was linear for the part of the motion being considered.

Extra Space for No.1

your name______________________________________

2. (15 pts) A mass $\boldsymbol{m}$ in a damped harmonic oscillator obeys the differential equation,

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0.$$

At a time $t = 0$ the particle is at a postion x_0 with zero velocity. Then, at $t = 0$, an additional force is added. The force is exponentially growing with time and has the form,

$$F(t) = F_0 e^{\lambda t}.$$

For large times, find the position as a function of time. Assume the time is so large that one can neglect any terms that die off exponentially with time. (You will NOT get full credit if you include the terms that die off!)

your name______________________________

Extra Space for No.2

Solution: Assume $\boldsymbol{x_p(t)}$ has the form $\boldsymbol{x_p = Ae^{\lambda}t}$. To find $\boldsymbol{A}$, insert into differential equation,

$$\begin{aligned}
(\boldsymbol{\lambda^2 + 2\beta\lambda + \omega_0^2})\boldsymbol{Ae^{\lambda t}} &= \boldsymbol{F_0 e^{\lambda t}}, \\
\boldsymbol{A} &= \frac{\boldsymbol{F_0}}{\boldsymbol{\lambda^2 + 2\beta\lambda + \omega_0^2}}, \\
\boldsymbol{x_p} &= \frac{\boldsymbol{F_0}}{\boldsymbol{\lambda^2 + 2\beta\lambda + \omega_0^2}}\boldsymbol{e^{\lambda t}}.
\end{aligned}$$

your name_________________________________

3. Bill the cat, who has mass $\boldsymbol{m}$, is dropped from a bridge with initial velocity $\boldsymbol{v_0 = 0}$. The bridge is a height $\boldsymbol{h}$ above the water. The drag force on Bill has magnitude $\boldsymbol{bAv^2}$, where $\boldsymbol{A}$ is Bill's cross sectional area and $\boldsymbol{b}$ is the drag coefficient. Give all answers in terms of $\boldsymbol{b, A, m, h}$ and the acceleration of gravity $\boldsymbol{g}$.

(a) (5 pts) What is Bill's terminal velocity, $\boldsymbol{v_t}$? (if he were dropped from an infinite height)

(b) (10 pts) Solve for Bill's velocity as a function of time. Give your answer in terms of $\boldsymbol{g}$, $\boldsymbol{v_t}$ and $\boldsymbol{t}$.

(c) (10 pts) When dropped from the bridge, how much time is required to strike the water? Give your answer in terms of $\boldsymbol{v_t, g}$ and $\boldsymbol{h}$

(d) (5 pts) If Bill was fed heavy marbles, then dropped again, would Bill take more or less time to strike the water. Justify your answer.

your name____________________________________

Extra Space for No.3

Solution:

a)

$$\begin{aligned} bAv_t^2 &= mg, \\ v_t &= \sqrt{\frac{mg}{bA}}. \end{aligned}$$

b) Defining downward as positive

$$\begin{aligned} \frac{dv}{dt} &= g - \frac{bAv^2}{m}, \\ gt &= \frac{dv}{1 - v^2/v_t^2}, \\ gt &= v_t \tanh^{-1}(v/v_t), \\ v &= v_t \tanh(gt/v_t). \end{aligned}$$

c)

$$\begin{aligned} h &= \int_0^t dt\ v(t) = v_t \int_0^t \tanh\left(\frac{gt}{v_t}\right), \\ \frac{gh}{v_t^2} &= \ln\left[\cosh\left(\frac{gt}{v_t}\right)\right], \\ e^{gh/v_t^2} &= \cosh\left(\frac{gt}{v_t}\right), \\ t &= \frac{v_t}{g}\cosh^{-1}\left(e^{gh/v_t^2}\right). \end{aligned}$$

d)

Less time. Drag force smaller relative to gravity, so he moves faster.

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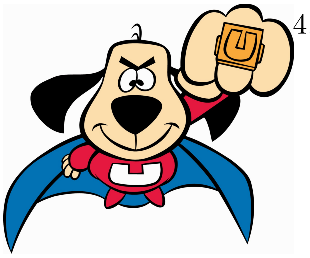

4. Underdog, who has mass M_0 is flying through the air with velocity $V_0\hat{x}$ when he encounters a stream of bullets headed directly his way. Each bullet has mass m_b and velocity $-v_b\hat{x}$. Underdog, being the great and gallant canine that he is, swallows all the bullets without harm. The bullets were fired by a machine gun at rest, which fires at a rate R_0 bullets per second. For questions involving the time t, $t = 0$ refers to the time Underdog encounters the first bullet.

(a) (5 pts) How many bullets does Underdog swallow before his velocity is zero? (If he never stops, just say so)

(b) (10 pts) What is Underdog's mass, $M(t)$, as a function of his velocity, $V(t)$? Give answer in terms of M_0, m_b, V_0, v_b and $V(t)$.

(c) (5 pts, Extra Credit) If Underdog's velocity is $V(t)$, how many bullets does Underdog swallow per second? Give answer in terms of R_0, v_b and $V(t)$.

(d) (10 pts, Extra Credit) Write a differential equation for $V(t)$ based on momentum conservation. The equation should include $V(t)$ and dV/dt, plus the constants M_0, m_b, V_0, v_b and R_0.

(e) (10 pts, Extra Credit) Solve for $V(t)$ in terms of the constants above.

your name____________________________________

Extra Space for No.4

Solution:

a) Let N_b refer to the number of bullets,

$$M_0V_0 - N_b m_b v_b = 0,$$
$$N_b = \frac{M_0V_0}{m_b v_b}.$$

b)

$$\begin{aligned} MV &= M_0V_0 - N_b m_b v_b, \\ N_b m_b &= M - M_0, \\ MV &= M_0V_0 - \frac{(M-M_0)}{m_b} m_b v_b, \\ MV &= M_0V_0 - Mv_b + M_0 v_b, \\ M &= \frac{M_0V_0 + M_0v_b}{V + v_b} \\ &= M_0\frac{V_0 + v_b}{V + v_b}. \end{aligned}$$

c)

The number of bullets per length in the air is

$$\rho_b = \frac{R_0}{v_b}$$

In a time Δt, Underdog moves a distance

$$\Delta X_U = V\Delta t,$$

In that time the length of the bullet stream swallowed by Underdog is

$$\begin{aligned} \Delta X_b &= v_b\Delta t + \Delta X_U \\ &= \Delta t(V + v_b), \end{aligned}$$

The number of bullets swallowed is

$$\begin{aligned} \Delta N_{\text{swallowed}} &= \rho_b \Delta X_b \\ &= \Delta t R_0 \frac{V + v_b}{v_b}, \\ \frac{dN_{\text{swallowed}}}{dt} &= R_0 \frac{V + v_b}{v_b}. \end{aligned}$$

d)

$$\begin{aligned} \frac{dP}{dt} &= -m_b v_b \frac{dN_{\text{swallowed}}}{dt} \\ &= -R_0 m_b (V + v_b), \\ M\frac{dV}{dt} + V\frac{dM}{dt} &= -R_0 m_b (V + v_b). \end{aligned}$$

From part (b)

$$M = M_0 \frac{V_0 + v_b}{V + v_b}.$$

Then,

$$\begin{aligned}
\frac{dM}{dt} &= -M_0 \frac{V_0 + v_b}{(V + v_b)^2} \frac{dV}{dt}, \\
M_0 \frac{V_0 + v_b}{V + v_b} \frac{dV}{dt} - V M_0 \frac{V_0 + v_b}{(V + v_b)^2} \frac{dV}{dt} &= -R_0 m_b (V + v_b), \\
\frac{dV}{dt} \frac{(V_0 + v_b)}{(V + v_b)^2} v_b &= -\frac{R_0 m_b}{M_0} (V + v_b).
\end{aligned}$$

e)

$$\int_0^V \frac{dV'}{(V' + v_b)^3} = -\frac{R_0 m_b}{M_0 (V_0 + v_b) v_b} t,$$

Define $\alpha \equiv R_0 m_b / (M_0 v_b (V_0 + v_b)$. Then, doing the integral,

$$\frac{\alpha t}{2} = \frac{1}{(V_0 + v_b)^2} - \frac{1}{(V + v_b)^2}.$$

Finally, solving for V,

$$\begin{aligned}
V &= -v_b + \frac{V_0 + v_b}{\sqrt{1 + \gamma t/2}}, \\
\gamma &= \alpha (V_0 + v_b)^2 = \frac{R_0 m_b (V_0 + v_b)}{M_0 v_b}.
\end{aligned}$$