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FYI: For the differential equation

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0,$$

the solutions are

$$\begin{aligned}
x &= A_1 e^{-\beta t}\cos\omega' t + A_2 e^{-\beta t}\sin\omega' t \quad \omega' = \sqrt{\omega_0^2 - \beta^2} \quad \text{(under damped)} \\
x &= A e^{-\beta t} + B t e^{-\beta t}, \quad \text{(critically damped)} \\
x &= A_1 e^{-\beta_1 t} + A_2 e^{-\beta_2 t}, \quad \beta_i = \beta \pm \sqrt{\beta^2 - \omega_0^2}, \quad \text{(over damped).}
\end{aligned}$$

Lagrange's equations:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) = \frac{\partial \mathcal{L}}{\partial q}.$$

Rotating frame:

$$m\frac{d\vec{v}}{dt} = \vec{F}_{\text{real}} - m\vec{\omega} \times (\vec{\omega} \times \vec{r}) - 2m\vec{\omega} \times \vec{v}.$$

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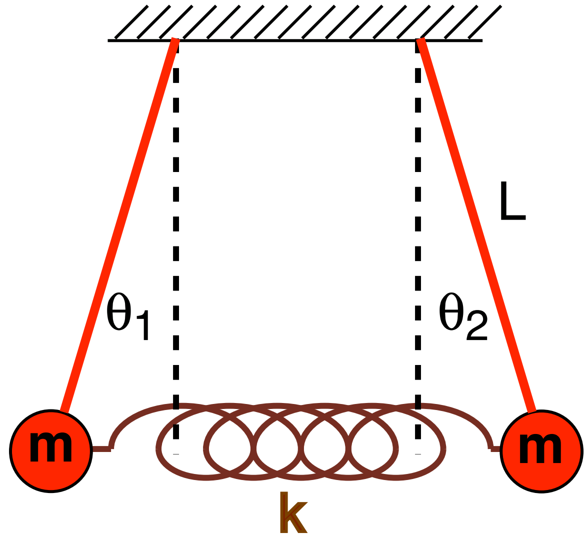

1. Consider two identical pendulums of length L with massless rods and weights of mass m that move in the plane of the paper as shown. They are connected by a spring of spring constant k. When the pendulums are both vertical, the spring is unstretched. Use the angular displacement of the two masses from their respective equilibrium positions as coordinates, θ_1 and θ_2. Choose $\theta_i > 0$ for movement to the right. Assume all displacements are small, i.e. all amplitudes are small.

   (a) (10 pts) Construct the Lagrangian.

   (b) (10 pts) Derive the equations of motion.

   (c) (20 pts) Find ω and A/B for solutions of the form,

$$\theta_1 = Ae^{i\omega t}, \quad \theta_2 = Be^{i\omega t}.$$

   There are two sets of solutions, so find A/B and ω for each set.

a) $L = \frac{m}{2}L^2\dot{\theta}_1^2 + \frac{m}{2}L^2\dot{\theta}_2^2 - \frac{1}{2}kL^2(\theta_1-\theta_2)^2 - \frac{m}{2}gL(\theta_1^2+\theta_2^2)$

b) $mL^2\ddot{\theta}_1 = -mgL\theta_1 - kL^2(\theta_1-\theta_2)$

$mL^2\ddot{\theta}_2 = -mgL\theta_2 - kL^2(\theta_2-\theta_1)$

$\ddot{\theta}_1 = -\frac{g}{L}\theta_1 - \omega_0^2(\theta_1-\theta_2)$ $\quad \boxed{\omega_0^2 \equiv k/m}$

$\ddot{\theta}_2 = -\frac{g}{L}\theta_2 - \omega_0^2(\theta_2-\theta_1)$

c) $-\omega^2 A = -\frac{g}{L}A - \omega_0^2(A-B)$

$-\omega^2 B = -\frac{g}{L}B - \omega_0^2(B-A)$

Divide l.h.s. by l.h.s = Divide r.h.s. by r.h.s

to eliminate ω^2

$$\frac{A}{B} = \frac{+(g/L)A + \omega_0^2(A-B)}{+(g/L)B + \omega_0^2(B-A)}$$

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Extra work space for #1

$$\frac{A}{B}\left(\frac{g}{L}+\omega_0^2\left(1-\frac{A}{B}\right)\right)=\frac{g}{L}\frac{A}{B}+\omega_0^2\left(\frac{A}{B}-1\right)$$

$$\left(\frac{A}{B}\right)^2\omega_0^2+\frac{A}{B}\left[\cancel{\frac{g}{L}}+\cancel{\omega_0^2}-\cancel{\frac{g}{L}}-\cancel{\omega_0^2}\right]=\omega_0^2$$

$$\frac{A}{B}=\pm 1$$

$$\mp\omega^2\left(\frac{A}{B}\right)=-\frac{g}{L}\left(\frac{A}{B}\right)-\omega_0^2\left(\frac{A}{B}\right)+\omega_0^2$$

for $\frac{A}{B}=1$, $-\omega^2=-\frac{g}{L}-\cancel{\omega_0^2}+\cancel{\omega_0^2}$

$$\omega^2=g/L$$

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for $\frac{A}{B}=-1$

$$\omega^2=+\frac{g}{L}+\omega_0^2+\omega_0^2$$

$$\omega^2=\frac{g}{L}+2\omega_0^2$$

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2. (20 pts) Nancy has a rocket traveling in deep space with speed v_0. The rocket's initial mass is M_0. Nancy fires a retro-rocket to slow down. The speed of the emitted gas relative to the rocket is v_g. What is the mass of Nancy's rocket when its speed reaches zero?

$$Mv = (M - \delta M)(v - \delta v) + \delta M (v + v_{gas})$$

$$0 = -\cancel{\delta M v} - M\delta v + \cancel{\delta M v} + \delta M v_{gas}$$

$$\frac{\delta M}{M} = \frac{\delta v}{v_{gas}}$$

Here $\delta M = -dM$ because $\delta M =$ mass lost

& $\delta v > 0$

$$-\int_{M_0}^{M_f} \frac{dM}{M} = \frac{1}{v_{gas}} \int_0^{v_0} dv$$

$$\ln \frac{M_f}{M_0} = \frac{-v_0}{v_{gas}}$$

$$M_f = M_0 e^{-v_0/v_{gas}}$$

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Extra work space for #2

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3. A particle of mass m moves in a harmonic oscillator potential,

$$V(x) = \frac{1}{2}m\omega_0^2 x^2,$$

and experiences a large drag force,

$$F_d = -2m\beta v,$$

which is **over-damped**. The particle then experiences an external force

$$f = F_0 \sin \omega t,$$

(a) (10 pts) Find $x(t)$ for large times (after all transients have faded away).

(b) (10 pts) The particle is placed at the origin with zero velocity at $t = 0$. Find the position as a function of time, $x(t)$, for all times $t > 0$.

a) $x_p = C e^{i\omega t}$

$m\ddot{x}_p + 2m\beta\dot{x}_p + m\omega_0^2 x_p = -iF_0 e^{i\omega t}$

↑ so that real part ~ sin ωt

$(-\omega^2 + 2i\omega\beta + \omega_0^2) C = -\frac{iF_0}{m}$

$$C = -i\left(\frac{F_0}{m}\right)\frac{1}{(-\omega^2+\omega_0^2) + 2i\omega\beta}$$

$$= -\frac{iF_0}{m}\frac{(\omega_0^2-\omega^2) - 2i\omega\beta}{(\omega^2-\omega_0^2)^2 + 4\omega^2\beta^2}$$

$$\mathrm{Re}\, C = \left(\frac{F_0}{m}\right)\frac{-2\omega\beta}{(\omega^2-\omega_0^2)^2 + 4\omega^2\beta^2}$$

$$\mathrm{Im}\, C = -\left(\frac{F_0}{m}\right)\frac{(\omega_0^2-\omega^2)}{(\omega^2-\omega_0^2)^2 + 4\omega^2\beta^2}$$

$$R\, x_p = x(t\to\infty) = \frac{F_0}{m}\frac{1}{(\omega^2-\omega_0^2)^2 + 4\omega^2\beta^2} \cdot \left\{-2\omega\beta\cos\omega t + (\omega_0^2-\omega^2)\sin\omega t\right\}$$

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Extra work space for #3

b) $\mathrm{Re}\, X_p = x(t \to \infty) = \frac{F_0}{m} \frac{1}{(\omega^2-\omega_0^2)^2 + 4\omega^2\beta^2} \cdot \left\{ -2\omega\beta \cos\omega t + (\omega_0^2 - \omega^2)\sin\omega t \right\}$

$= \alpha \left\{ -2\omega\beta\cos\omega t + (\omega_0^2-\omega^2)\sin\omega t \right\}$

$\alpha \equiv \frac{F_0}{m} \frac{1}{(\omega^2-\omega_0^2)^2 + 4\omega^2\beta^2}$

$x = A e^{-\gamma_1 t} + B e^{-\gamma_2 t} + x_p$, γ_1, γ_2 defined on Eq. sheet.

$0 = A + B - 2\omega\beta\alpha = x(t=0)$

$0 = -\gamma_1 A - \gamma_2 B + \alpha\omega(\omega_0^2 - \omega^2) = \dot{x}(t=0)$

$0 = -\gamma_1 (B - 2\omega\beta\alpha) - \gamma_2 B + \alpha\omega(\omega_0^2-\omega^2)$

$B = \frac{2\omega\beta\alpha\gamma_1 - \alpha\omega(\omega_0^2-\omega^2)}{\gamma_1 - \gamma_2}$

$A = \frac{2\omega\beta\alpha\gamma_2 - \alpha\omega(\omega_0^2-\omega^2)}{\gamma_2 - \gamma_1}$

$x = A e^{-\gamma_1 t} + B e^{-\gamma_2 t} + \alpha \left\{ -2\omega\beta\cos\omega t + (\omega_0^2-\omega^2)\sin\omega t \right\}$

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4. A particle of mass m is aimed at a heavy spherical target of radius R, located at the origin. The particle, when far away has kinetic energy E_0, and when approaching the target feels a spherically-symmetric potential,

$$V(r) \quad = \quad -V_0 e^{-(r-R)/\lambda}.$$

(a) (5 pts) Which of following quantities are conserved? (remain constant throughout any and all trajectories while $r > R$) Circle all correct answers.

- The particle's total energy
- The particle's kinetic energy
- The particle's potential energy
- The particle's momentum component, p_x
- The particle's momentum component, p_y
- The particle's momentum component, p_z
- The magnitude of the particle's momentum, $|\vec{p}|$
- The particle's radial velocity, $v_r = \hat{r} \cdot \vec{v}$
- The magnitude of the particle's tangential velocity, $v_t = |\hat{r} \times \vec{v}|$
- The particle's angular momentum component, L_x
- The particle's angular momentum component, L_y
- The particle's angular momentum component, L_z
- The magnitude of the particle's angular momentum vector, $|\vec{L}|$.

(b) (15 pts) Solve for the cross section for a collision with the target in terms of R, m, V_0, λ and E_0.

Angular Momentum conservation

$$R^2 P_f^2 = P_0^2 b^2, \quad P^2 = 2mT$$

$$R^2 \left(2m\left(E + V_0\right)\right) = 2mE\, b^2$$

$$b^2 = \left(\frac{E + V_0}{E}\right) R^2$$

$$\sigma = \pi b^2 = \pi R^2 \left(\frac{E + V_0}{E}\right)$$

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Extra work space for #4