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FYI: For the differential equation

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0,$$

the solutions are

$$\begin{aligned}
x &= A_1 e^{-\beta t}\cos\omega' t + A_2 e^{-\beta t}\sin\omega' t \quad \omega' = \sqrt{\omega_0^2 - \beta^2} \quad \text{(under damped)}\\
x &= A e^{-\beta t} + B t e^{-\beta t}, \quad \text{(critically damped)}\\
x &= A_1 e^{-\beta_1 t} + A_2 e^{-\beta_2 t}, \quad \beta_i = \beta \pm \sqrt{\beta^2 - \omega_0^2}, \quad \text{(over damped).}
\end{aligned}$$

Lagrange's equations:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) = \frac{\partial \mathcal{L}}{\partial q}.$$

Rotating frame:

$$m\frac{d\vec{v}}{dt} = \vec{F}_{\rm real} - m\vec{\omega}\times(\vec{\omega}\times\vec{r}) - 2m\vec{\omega}\times\vec{v}.$$

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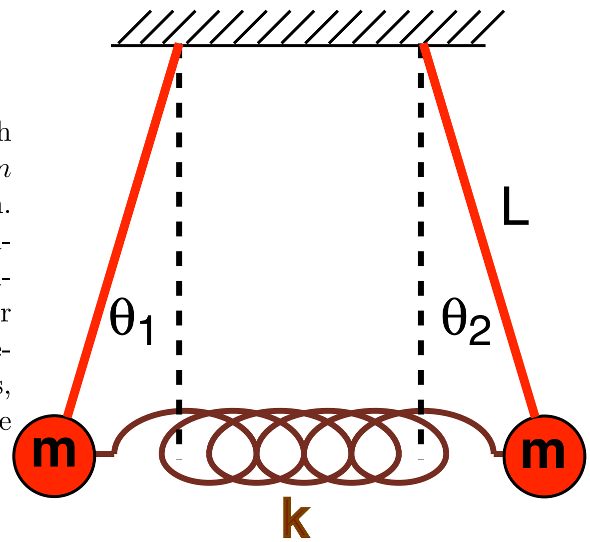
1. Consider two identical pendulums of length L with massless rods and weights of mass m that move in the plane of the paper as shown. They are connected by a spring of spring constant k. When the pendulums are both vertical, the spring is unstretched. Use the angular displacement of the two masses from their respective equilibrium positions as coordinates, θ_1 and θ_2. Choose $\theta_i > 0$ for movement to the right.



(a) (15 pts) Construct the Lagrangian.
(make small angle approximation, i.e. $\mathcal{L}$ only keeps terms quadratic in θ_i or $\dot{\theta}_i$)

(b) (10 pts) Derive the equations of motion.

a) $V \cong mgL(1-\cos\theta_1 + 1 - \cos\theta_2) + \frac{1}{2}kL^2(\theta_1-\theta_2)^2$

$\cong \frac{1}{2}mgl(\theta_1^2+\theta_2^2) + \frac{1}{2}kL^2(\theta_1-\theta_2)^2$

$L = \frac{1}{2}mL^2(\dot{\theta}_1^2+\dot{\theta}_2^2) - mgL(\theta_1^2+\theta_2^2) - \frac{1}{2}kL^2(\theta_1-\theta_2)^2$

b) $\frac{d}{dt}(\partial\mathcal{L}/\partial\dot{\theta}_1) = \frac{\partial\mathcal{L}}{\partial\theta_1}$

$mL^2\ddot{\theta}_1 = -mgL\theta_1 - kL^2(\theta_1-\theta_2)$

$mL^2\ddot{\theta}_2 = -mgL\theta_2 - kL^2(\theta_2-\theta_1)$

$\ddot{\theta}_1 = -\frac{g}{L}\theta_1 - \frac{k}{m}(\theta_1-\theta_2)$

$\ddot{\theta}_2 = -\frac{g}{L}\theta_2 - \frac{k}{m}(\theta_2-\theta_1)$

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Extra work space for #1

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2. (25 pts) A particle of mass m moves according to a potential

$$V(x, y, z) = \frac{k}{2}(x^2 + z^2).$$

Circle all conserved quantities. ($\vec{p}$ refers to momentum, $\vec{L}$ to angular momentum, E is the total energy)

- p_x
- p_y
- p_z
- $p_x + p_y$
- $p_x + p_z$
- $p_y + p_z$
- L_x
- L_y
- L_z
- $L_x + L_y$
- $L_x + L_z$
- $L_y + L_z$
- E

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3. (25 pts) A mass is dropped from a height h above the equator. Find the deflection, the position where the mass lands relative to being directly below the dropping point, due to the Coriolis force. Give answer in terms of h, g and the angular velocity of Earth's rotation ω, and assume h is not so high that g changes.

   SCORING: magnitude (20 pts), direction, i.e.north, east, west or south (5 pts). At least 10 points will be subtracted if your answer for the magnitude of the deflection does not have dimensions of length.

$\vec{\omega} = \omega \hat{y}$ (north)

$\vec{v} = -gt\hat{z}$

$a_{coriolis} = -2\vec{\omega} \times \vec{v} = -2\omega g t \hat{x}$ (east :)

$v_x = -2\omega g \int_0^t t' dt' = -\omega g t^2$

$x = -\frac{1}{3}\omega g t^3$, $h = \frac{1}{2} g t^2$

$t = \sqrt{2h/g}$

$x = -\frac{1}{3}\omega g \frac{(2h)^{3/2}}{g^{3/2}}$

$= -\frac{1}{3}\omega \frac{(2h)^{3/2}}{g^{1/2}}$

$\boxed{\text{east}}$

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Extra work space for #3

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4. A particle of mass m is in a circular orbit or radius R, moves according to a potential

$$V = -\frac{V_0}{r^{3/2}},$$

where $V_0 > 0$.

(a) (5 pts) What is the particle's speed while in a circular orbit? Give answer in terms of V_0, m and R.

(b) (5 pts) What is the particle's angular momentum, L, in the circular orbit? Give answer in terms of V_0, m and R.

(c) (5 pts) For a particle with this angular momentum, what is the effective radial potential? Provide a sketch. Label the radius of the circular orbit.

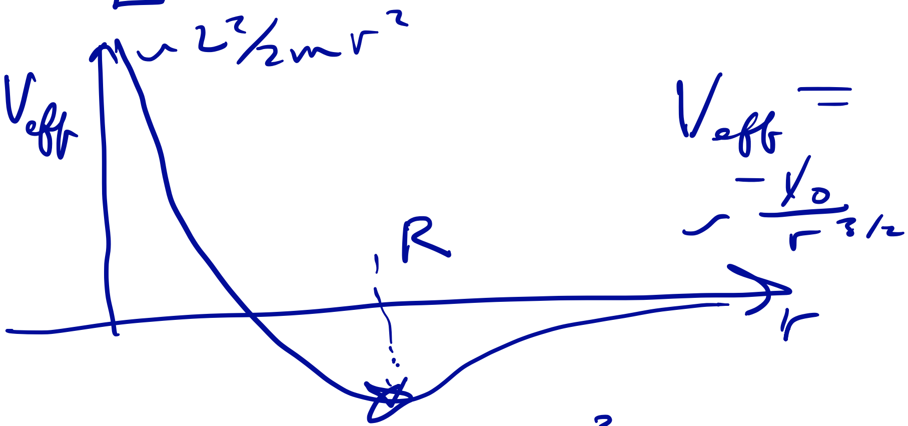
(d) (10 pts) What is the angular frequency of small oscillations of the radial distance r for such a particle with angular momentum L about the circular orbit? Give answer in terms of V_0, m and R.

a) $F = -\frac{3}{2}\frac{V_0}{r^{5/2}} = -m\frac{v^2}{r}$

$$v = \sqrt{\frac{3}{2m}\frac{V_0}{R^{3/2}}}$$

b) $L = mvR = \sqrt{\frac{3\,V_0 R^{1/2} m}{2}}$

c)



$$V_{eff} = \frac{L^2}{2mr^2} - \frac{V_0}{r^{3/2}}$$

d) $k = \frac{d^2V}{dr^2} = 3\frac{L^2}{mR^4} - \frac{15}{4}\frac{V_0}{R^{7/2}} = \frac{3\,V_0}{4\,R^{7/2}}$

$$\omega = \sqrt{k/m} = \left(\frac{3\,V_0}{4\,m}\right)^{1/2}\cdot\frac{1}{R^{7/4}}$$

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Extra work space for #4