your name____________________________________

Physics 321 Midterm #1 - Wednesday, Oct. 5, 2022

Some integrals:

$$\begin{aligned}
\int \frac{dx}{1+x^2} &= \tan^{-1}(x), \\
\int \frac{dx}{1-x^2} &= \tanh^{-1}(x), \\
\int \frac{dx}{\sqrt{1-x^2}} &= \sin^{-1}(x), \\
\int \frac{dx}{\sqrt{1+x^2}} &= \sinh^{-1}(x), \\
\int dx\ \tan(x+\phi) &= -\ln(\cos(x+\phi)) \\
\int dx\ \tanh(x+a) &= \ln(\cosh(x+a)).
\end{aligned}$$

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80 points possible

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1. Aliyah and her iceboat have a mass M_0 and are gliding on a frictionless lake straight toward a long line penguins with initial speed v_0. Each penquin has a mass m, and there are λ penguins per unit length. The penguins jump straight up (the only direction they can jump due to the frictionless surface) as the boat approaches and land on the boat.

   (a) (5 pts) What is the momentum of the boat, including Aliyah and her penquin passengers, as a funtion of time t? The time is measured from the moment she first reaches the line of penguins.

   (b) (5 pts) What is Aliyah's velocity as a function of the distance x? The distance is measured from the point she first reaches the penguins.

   (c) (10 pts) Find Aliyah's position as a function of time.

Solution:
a) Momentum doesn't change=$M_0 v_0$.
b) Mass as function of distance, $M = M_0 + m\lambda x$.

$$v(t) = \frac{M_0 v_0}{M_0 + m\lambda x}$$

c)

$$\begin{aligned}
\frac{dx}{dt} &= \frac{M_0 v_0}{M_0 + m\lambda x}, \\
t &= \int \frac{dx}{dx/dt} \\
&= \frac{1}{M_0 v_0} \int dx\ (M_0 + m\lambda x) \\
&= \frac{1}{v_0}\left(x + \frac{m\lambda x^2}{2M_0}\right), \\
\frac{m\lambda}{2M_0 v_0} x^2 + \frac{1}{v_0} x - t &= 0, \\
x &= \frac{M_0}{m\lambda}\left\{-1 + \sqrt{(1 + 2m\lambda t v_0/M_0}\right\}
\end{aligned}$$

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2. A particle of mass m and charge q moves in the $x-y$ plane under the influence of a magnetic field $\vec{B} = B\hat{z}$. At $t = 0$ the particle is at the origin with velocity $v_x(t = 0) = v_0$ and $v_y(t = 0) = 0$.

   (a) (15 pts) Solve for the velocity, both v_x and v_y, as a function of time.

   (b) (10 pts) Solve for the position as a function of time.

Solution:
a) Choose phase and magnitude such that $v_x(t = 0) = v_0$

$$\begin{aligned}
\vec{F} &= q\vec{v} \times \vec{B}, \\
\partial_t v_x &= qBv_y/m, \\
\partial_t v_y &= -qBv_x/m, \\
v_x(t) &= v_0 \cos(\omega t), \\
v_y &= -v_0 \sin(\omega t), \\
\omega &= \frac{qB}{m}.
\end{aligned}$$

b) Integrate

$$\begin{aligned}
x(t) &= \frac{v_0}{\omega} \sin(\omega t), \\
y(t) &= -\frac{v_0}{\omega}(\cos(\omega t) - 1).
\end{aligned}$$

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3. Two bicyclists of the same size and shape, with the same drag coefficient, travel down the road. Cyclist A travels at 15 mph and cyclist B travels at 20 mph. Assume all the work goes into fighting air resistance (force scales as v^2).

   (a) (5 pts) If cyclist A requires 150 Watts to maintain his speed, how much power does cyclist B require?

   (b) (5 pts) If both cyclists travel the same distance, and if Cyclist A expends a net energy of 1 kilowatt-hour due to the biking, how much energy does Cyclist B expend?

Solution:
a) $F \propto v^2, \quad P \propto v^3.$

$$\frac{P_B}{P_A} = \frac{20^3}{15^3},$$
$$P_B = 150\frac{20^3}{15^3} = 355.6\text{W}.$$

b) $E_B = (4/3)^2 E_A$

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4. Bill the cat, who has mass m, is fired straight upward from a cannon with muzzle velocity v_0. The drag force on Bill has magnitude bAv^2, where A is the cross sectional area. Assume $v_0 < v_t$, where v_t is the terminal velocity.

(a) (5 pts) What is the terminal velocity? Give answer in terms of b, A, m and the acceleration of gravity g.

(b) (10 pts) Solve for Bill's velocity as a function of time on the way up. Give your answer in terms of v_0, g and v_t.

(c) (5 pts) How much time is required to reach the top of the trajectory?

(d) (5 pts) If, after landing on his feet, Bill was fed heavy marbles, then fired from the cannon again, with the same initial muzzle velocity, would Bill go higher? or lower? – than on his first trip. Justify your answer.

(e) (5 pts, extra credit) What is the maximum height of Bill's trajectory?

(f) (5 pts, extra credit, contingent on previous answer being correct) When Bill passes the cannon on the way down, is his speed greater, or less than, or equal to v_0? Justify your answer.

Solution:
a)

$$\frac{dv}{dt} = -g - bAv^2/m.$$

Terminal velocity is when $dv/dt = 0$,

$$v_t = \sqrt{mg/bA}$$

b)

$$\begin{aligned}
\frac{dv}{dt} &= -g(1 + v^2/v_t^2), \\
gt &= -\int_{v_0}^{v} \frac{dv'}{1 + v'^2/v_t^2}, \\
&= -v_t[\tan^{-1}(v/v_t) - \tan^{-1}(v_0/v_t)], \\
v &= v_t \tan\left[\tan^{-1}(v_0/v_t) - gt/v_t\right].
\end{aligned}$$

c) Top of trajectory when $v = 0$ and

$$\begin{aligned}
\tan^{-1}(v_0/v_t) - gt_{\rm top}/v_t &= 0, \\
t_{\rm top} = (v_t/g)\tan^{-1}(v_0/v_t)
\end{aligned}$$

Note that for $v_0 << v_t$ that $t \approx v_0/g$, the expected result for no air resistance
d) higher – air resistance is less important because drag force/mass and v_t are higher.

e)

$$
\begin{aligned}
h &= \int_0^{t_{\rm top}} dt\ v_t \tan\left[\tan^{-1}(v_0/v_t) - gt/v_t\right], \\
&= -\frac{v_t^2}{g}\left\{\ln\left(\cos(\tan^{-1}(v_0/v_t) - gt_{\rm top}/v_t)\right)\ln\left(\cos(\tan^{-1}(v_0/v_t)\right)\right\} \\
&= -\frac{v_t^2}{g}\ln\left(\cos(\tan^{-1}(v_0/v_t)\right).
\end{aligned}
$$

f) Recalculate everything from the top of the trajectory. Let's define positive as moving downward with $y = 0$ being the top and $y = h$. The equations then look the same, except g is negative. After some work,

$$
h = \frac{v_t^2}{g}\ln\left\{\cosh[\tanh^{-1}(v_f/v_t)]\right\}
$$

where v_f is the Bill's speed when he hits the ground.
Use the facts that

$$
\begin{aligned}
\cos(\tan^{-1}(x)) &= \frac{1}{\sqrt{1+x^2}}, \\
\cosh(\tanh^{-1}(x)) &= \frac{1}{\sqrt{1-x^2}}.
\end{aligned}
$$

Equating the two expressions for h,

$$
1 + (v_0/v_t)^2 = \frac{1}{1-(v_f/v_t)^2},
$$

then performing a Taylor expansion,

$$
1 + (v_0/v_t)^2 = 1 + (v_f/v_t)^2 + (v_f/v_t)^4 + (v_f/v_t)^6 + \cdots
$$

So, clearly v_f must be less than v_0.