your name________________________________

Physics 321 Exam #1 - Monday, Feb. 18

FYI: For the differential equation

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0,$$

the solutions are

$$\begin{aligned}
x &= A_1 e^{-\beta t}\cos\omega' t + A_2 e^{-\beta t}\sin\omega' t \quad \omega' = \sqrt{\omega_0^2 - \beta^2} \quad \text{(under damped)} \\
x &= Ae^{-\beta t} + Bte^{-\beta t}, \quad \text{(critically damped)} \\
x &= A_1 e^{-\beta_1 t} + A_2 e^{-\beta_2 t}, \quad \beta_i = \beta \pm \sqrt{\beta^2 - \omega_0^2}, \quad \text{(over damped).}
\end{aligned}$$

Fourier expansion ($\omega = 2\pi/\tau$):

$$\begin{aligned}
F(t) &= \frac{f_0}{2} + \sum_{n>0} f_n \cos(n\omega t) + g_n \sin(n\omega t), \\
f_n &= \frac{2}{\tau}\int_{-\tau/2}^{\tau/2} dt\ F(t)\cos(n\omega t), \\
g_n &= \frac{2}{\tau}\int_{-\tau/2}^{\tau/2} dt\ F(t)\sin(n\omega t).
\end{aligned}$$

Some integrals:

$$\begin{aligned}
\int \frac{dx}{1+x^2} &= \tan^{-1}(x), \\
\int \frac{dx}{1-x^2} &= \tanh^{-1}(x), \\
\int \frac{dx}{\sqrt{1-x^2}} &= \sin^{-1}(x), \\
\int \frac{dx}{\sqrt{1+x^2}} &= \sinh^{-1}(x),
\end{aligned}$$

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1. After being dropped with zero initial velocity, a solid copper ball of mass m falls with a drag force $\gamma A v^2$, where A is the cross sectional area. The magnitude of the gravitational acceleration is g.

   (a) (5 pts) What is the terminal velocity?

   $$mg = \gamma A v_t^2 , \quad v_t = \sqrt{mg/\gamma A}$$

   (b) (10 pts) Solve for the speed as a function of time.

   $$\frac{dv}{dt} = -g + \frac{\gamma A}{m} v^2 = -g + g\frac{v^2}{v_t^2}$$

   $$\int_0^v \frac{dv'}{\left(1 - \frac{v'^2}{v_t^2}\right)} = -gt$$

   $$v_t \tanh^{-1}(v/v_t) = -gt$$

   $$v = -v_t \tanh gt/v_t$$

   (c) (5 pts) If two solid copper balls A and B are dropped simultaneously from a large height, one with $R_B > R_A$. They are both affected by air resistance, with the drag forces proportional to their cross-sectional areas. Circle the true statement below:

   - Ball A falls faster than B
   - Ball B falls faster than A
   - The balls fall with the same speeds.

your name____

2. Ted and his iceboat have a combined mass of M_0. Ted's boat slides without friction on top of a frozen lake. Ted's boat has a winch and he wishes to wind up a long heavy rope, which has the same mass M_0, and length L_0. The rope is laid out in a straight line on the ice. Ted's boat starts at rest at one end of the rope, then brings the rope on board at a constant length per time of w. Clearly express all answers in term of M_0, L_0 and w.

   (a) (5 pts) Before Ted turns on the winch, what is the position of the center of mass relative to the boat?

   (b) (5 pts) Immediately after the rope is entirely on board, what is Ted's displacement relative to his original position?

   (c) (10 pts) Find Ted's velocity as a function of time.

a) $x_{cm} = \dfrac{M_0 \cdot \emptyset + M_0 \, L_0/2}{2M_0} = \dfrac{L_0}{4}$

b) $L_0/4$

c) $\left(M_0 + \dfrac{wt}{L_0} M_0\right) v_{ted} = \left(M_0 - \dfrac{wt}{L_0} M_0\right)\left(v_w - v_{ted}\right)$

$v_{ted} = \dfrac{1 - \frac{wt}{L_0}}{2} v_w$

your name____

3. (20 pts) A particle of mass m is connected to a spring with spring constant k. Damping is added, proportional to the velocity, and adjusted so that the damping is critical. After being at rest for a long time, an impulse I is applied to the particle. The force from the impulse is:

$$F(t) = I\delta(t).$$

Find the position relative to the equilibrium position, $x(t)$, for $t > 0$.

$$v_0 = I/m, \quad x_0 = 0$$

$$x = Ae^{-\beta t} + Bte^{-\beta t}, \quad \beta = \sqrt{k/m}$$

$$0 = A$$

$$\frac{I}{m} = B$$

$$x = \frac{I}{m} t e^{-\beta t}$$

your name____________________________________

4. Consider the periodic force

$$F(t) = \begin{cases} 0, & -\tau/2 < t < -\tau/4 \\ F_0, & -\tau/4 < t < \tau/4 \\ 0, & \tau/4 < t < \tau/2 \end{cases}$$

If the force is expressed as a Fourier decomposition,

$$F(t) = \frac{f_0}{2} + \sum_{n>0} f_n \cos(n\omega t) + g_n \sin(n\omega t), \quad \omega = 2\pi/\tau,$$

(a) (5 pts) which coefficients f_n and g_n are non-zero?

$f_0, f_1, f_3, f_5 \ldots\ldots$

(b) (15 pts) Find the coefficients.

$$f_0 = \frac{2}{\tau}\int_{-\tau/4}^{\tau/4} dt\, F_0 = F_0$$

$$f_{n=1,3,5,7} = \frac{4F_0}{\tau}\int_0^{\tau/4} dt \cos n\omega t, \quad \omega = \frac{2\pi}{\tau}$$

$$= \frac{4F_0}{\tau}\frac{1}{n(2\pi/\tau)} \cdot \sin\frac{n\pi}{2}$$

$$= \frac{2F_0}{n\pi}\sin(n\pi/2)$$

$$= \begin{cases} \frac{2F_0}{n\pi}, & n = 1, 5, 9, 13, \ldots \\ -\frac{2F_0}{2\pi}, & n = 3, 7, 11, 15 \ldots \end{cases}$$