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Physics 321 Exam #1 - Wednesday, Oct 9

FYI: For the differential equation

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0,$$

the solutions are

$$\begin{aligned}
x &= A_1 e^{-\beta t}\cos\omega' t + A_2 e^{-\beta t}\sin\omega' t \quad \omega' = \sqrt{\omega_0^2 - \beta^2} \quad \text{(under damped)}\\
x &= Ae^{-\beta t} + Bte^{-\beta t}, \quad \text{(critically damped)}\\
x &= A_1 e^{-\beta_1 t} + A_2 e^{-\beta_2 t}, \quad \beta_i = \beta \pm \sqrt{\beta^2 - \omega_0^2}, \quad \text{(over damped).}
\end{aligned}$$

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1. After being dropped with zero initial velocity, a solid copper ball of mass m falls with a drag force $\gamma\rho_m A v^2$, where A is the cross sectional area, γ is the drag coefficient and ρ_m is the density of air. The magnitude of the gravitational acceleration is g.

   (a) (4 pts) Solve for the speed as a function of time.

   (b) (2 pts) Three spherical balls, A, B and C are dropped from the same height. All three have the same radius R. A has mass m with the mass uniformly distributed in the sphere. B has mass $2m$, and also with the mass uniformly distributed. C has mass m but is hollow in the center and has most of the mass distributed on the outside. List the balls in order of which ones reach the ground first. Note if any balls hit at the same time.

   Solution: a)

$$\begin{aligned}
m\frac{dv}{dt} &= -\gamma\rho_m A v^2 + mg, \ \text{down is positive}\\
t &= \int_0^v dv' \, \frac{1}{g - \gamma\rho_m A v'^2/m}, \quad \text{do trig sub}\\
t &= \frac{1}{g}\tanh^{-1}(v/v_c), \quad v_c = \sqrt{\frac{mg}{\gamma\rho_m A}}, \quad v_c \text{ is critical vel.}\\
v(t) &= v_c \tanh(gt/v_c).
\end{aligned}$$

   b) ~~A is fastest, B and C are the same.~~

   b) B is fastest, A & C are the same

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2. (4 pts) Ted and Ned have identical iceboats of mass M (including their own masses) which slide on a frictionless frozen lake. They begin at rest separated by a distance L connected by a taut rope of length L and mass m. Both Ted and Ned have winches, which begin winding the rope onto their boats and pulling them together. Each winch collects a length of rope per time of w onto their boat. What is Ted's speed just before the boats collide?

   Solution: Boats move with equal and opposite velocity. Rope doesn't have any velocity. $v_t - v_r = w$, so $v_{\text{Ted}} = 2$, and by symmetry, $v_{\text{Ned}} = -w$. Speeds are constant and $v_{\text{Ted}} = w$ until they actually collide.

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3. A particle of mass m moves according to the potential

$$V(x) = \begin{cases} -\alpha x^4, & x > 0 \\ \alpha x^4, & x < 0 \end{cases} \qquad \alpha > 0.$$

A particle begins at position x_0 with zero total energy (potential plus kinetic).

(a) (2 pt) Sketch the potential

(b) (3 pts) How much time is required to move to position x_f?

(c) (2 pt) If the particle's initial velocity is toward the origin, what is the minimum position of its trajectory?

(d) (1 pt) How much time is required to reach that position?

Solution:

a)

b)

$$\begin{aligned} \int_0^t dt' &= \int_{x_0}^{x_f} \frac{dx'}{\sqrt{(2/m)(E + \alpha x'^4)}} \\ &= \sqrt{\frac{m}{2\alpha}} \int_{x_0}^{x_f} \frac{dx'}{x'^2} \\ &= \sqrt{\frac{m}{2\alpha}} \left(\frac{1}{x_f} - \frac{1}{x_0} \right). \end{aligned}$$

c) Minimum is at $x = 0$

d) Never gets there, $1/(x \to 0) \to \infty$.

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4. Consider a particle of mass m moving in a potential

$$V(x) = -V_0 \cos(x/\lambda), \quad V_0 > 0.$$

The particle also feels a small damping force $-bv$.

(a) (2 pts) Sketch the potential

(b) (3 pts) Find the angular frequency of small oscillations of a particle about the minimum of the potential in the limit $b \to 0$.

(c) (2 pts) Imagine a particle which is initially at rest at position a, where $a \ll \lambda$. Find the position of the particle as a function of time. For this part assume b is small but not negligibly small.

Solution:

a)

b)

$$\begin{aligned} k_{\rm eff} &= \left.\frac{dV(x)}{dx}\right|_{x=0} && (1)\\ &= \frac{V_0}{\lambda^2}, && (2)\\ \omega_0 &= \sqrt{\frac{k_{\rm eff}}{m}} = \frac{1}{\lambda}\sqrt{\frac{V_0}{m}}. && (3) \end{aligned}$$

c) Just copy from given eq. at beginning of exam:

$$\begin{aligned} x &= A_1 e^{-\beta t}\cos\omega' t + A_2 e^{-\beta t}\sin\omega' t \quad \omega' \equiv \sqrt{\omega_0^2 - \beta^2},\ \beta \equiv \frac{b}{2m},\\ a &= A_1,\\ 0 &= -\beta A_1 + \omega' A_2,\\ A_1 &= A, \quad A_2 = \frac{\beta a}{\omega'}. \end{aligned}$$