FEMTOSCOPY

Scott Pratt, June 2020

Motivation

- Experiments only measure momenta
  — But dynamics involve space and time
- Eq. of state and collective flow affect dynamics
  — To infer EoS, must test dynamics
  — Liquid ⇒ slow emission (evaporation)
  — Gas ⇒ rapid emission (explosion)
- Entropy inference requires volume
  — Femtoscopy provides volume for phase space density

THEORY

Koonin Eq.

$$\vec{p} = \vec{p}_1 + \vec{p}_2)/2, \quad \vec{q} = (\vec{p}_1 - \vec{p}_2)/2, \quad \vec{r} = \vec{r}_1 - \vec{r}_2$$

$$C(\vec{p}, \vec{q}) = \frac{N(\vec{p}_1, \vec{p}_2)}{N(\vec{p}_1)N(\vec{p}_2)}$$

$$= \int d^3r \; S(\vec{p}, \vec{r})|\phi_q(\vec{r})|^2,$$

relative wave function

$$S(\vec{p}, \vec{r}) = \frac{\int d^3r_1 d^3r_2 \; f(\vec{p}_1, \vec{r}_1, t) f(\vec{p}_2, \vec{r}_2, t) \delta(\vec{r} - [\vec{r}_1 - \vec{r}_2])}{\int d^3r_1 d^3r_2 \; f(\vec{p}_1, \vec{r}_1, t) f(\vec{p}_2, \vec{r}_2, t)}$$

"source" function:

— integrates to unity

— asymptotic probability of being separated by *r* (2 particles of same velocity)

— measured size/shape of phase space cloud (for fixed *p*), not source

GOAL: Determine *S(p,r)* from measurement of *C(p,q)*

THEORY

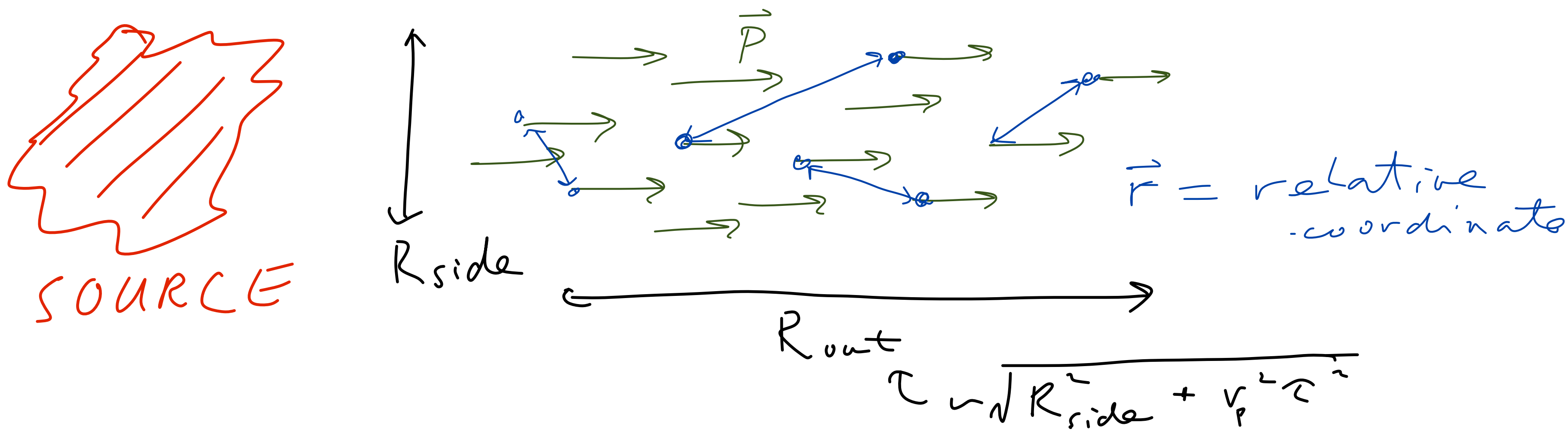
Visually,

$$\vec{p} = \vec{p}_1 + \vec{p}_2)/2, \quad \vec{q} = (\vec{p}_1 - \vec{p}_2)/2, \quad \vec{r} = \vec{r}_1 - \vec{r}_2$$

$$C(\vec{p},\vec{q}) = \frac{N(\vec{p}_1,\vec{p}_2)}{N(\vec{p}_1)N(\vec{p}_2)}$$

$$= \int d^3x_1 d^3x_2 \; S(\vec{p},\vec{r})|\phi_q(\vec{r})|^2,$$

$$S(\vec{p},\vec{r}) = \frac{\int d^3r_1 d^3r_2 \; f(\vec{p}_1,\vec{r}_1,t)f(\vec{p}_2,\vec{r}_2,t)\delta(\vec{r}-[\vec{r}_1-\vec{r}_2])}{\int d^3r_1 d^3r_2 \; f(\vec{p}_1,\vec{r}_1,t)f(\vec{p}_2,\vec{r}_2,t)}$$



THEORY

Deriving the Koonin Equation

$$P(\vec{p}) = \sum_F \left| \int d^4x \; T_F(x) e^{-ip\cdot x} \right|^2 \delta(E_F + E_p - E_{\text{tot}})$$

Approximation #1
Modify if > 2 identical particles

$$P(\vec{p}_1, \vec{p}_2) = \sum_F \left| \int d^4x_1 d^4x_2 \; T_F(x_1, x_2) U(x_1, x_2; \vec{p}_1, \vec{p}_2, t \to \infty) \right|^2 \delta(E_F + E_1 + E_2 - E_{\text{tot}})$$

x_1,x_2 are last points of interaction with remainder *F*.
what about Coulomb interaction with source?

Approximation #2

$$U(x_1, x_2; \vec{p}_1, \vec{p}_2) = e^{-iP\cdot(x_1+x_2)} \phi_{\vec{q}}(\vec{r}')$$

r′ is relative position in c.o.m. frame

relative wave function defined for equal times

This approximation not needed for pure identical particle interference

THEORY

Deriving the Koonin Equation

Approximation #3

$$\sum_F \to \sum_{F,F'},$$

$$T_F(x_1, x_2) \to T_F(x_1) T_{F'}(x_2)$$

Factorization of *T*-matrix: Emissions independent if outgoing wave function is plane wave

Otherwise, correlation would not be unity for non-interacting non-identical particles

THEORY

Deriving the Koonin Equation

Approximation #4

True for thermal emission
Good for small *q*

$$s_a(p,x) \equiv \sum_F \int d^4\delta x e^{-ip\cdot\delta x} T_F^*(x+\delta x/2) T_F(x-\delta x/2)\delta(E_F+E_p-E_a),$$

$$\begin{aligned} s_a(p_a,x_a)s_b(p_b,x_b)|_{p_{a0}+p_{b0}=E_a+E_b} &= s_a(E_a,\vec{p}_a)s_b(E_b,x_b) \\ &= s_a([E_a+E_b]/2,x_a)s_b([E_a+E_b]/2,x_b) \end{aligned}$$

"Smoothness approximation"

THEORY

Variants of Koonin Equation

Approximation #4

$$S(\vec{p},\vec{r}) = \frac{\int d^3r_1 d^3r_2\ f(\vec{p}_1,\vec{r}_1,t)f(\vec{p}_2,\vec{r}_2,t)\delta(\vec{r}-[\vec{r}_1-\vec{r}_2])}{\int d^3r_1 d^3r_2\ f(\vec{p}_1,\vec{r}_1,t)f(\vec{p}_2,\vec{r}_2,t)}$$

$$\approx \frac{\int d^3r_1 d^3r_2\ f(\vec{p},\vec{r}_1,t)f(\vec{p},\vec{r}_2,t)\delta(\vec{r}-[\vec{r}_1-\vec{r}_2])}{\int d^3r_1 d^3r_2\ f(\vec{p},\vec{r}_1,t)f(\vec{p},\vec{r}_2,t)},$$

$$\vec{p} = \frac{\vec{p}_1+\vec{p}_2}{2}$$

Equal within smoothness approximation

True for thermal emission
Good for small *q*
sometimes numerator & denominator treated differently

For large sources (many thermal wavelengths), approximations #3 and #4 should be good

Difficult to estimate accuracy for *pp* or e^+e^- collisions

THEORY

Accuracy of Koonin Equation

Approximation #1 good when $f(p,r,t)$<<1 and not too close to Coulomb barrier
— be careful with pions at low p_t
— be careful with E is close to Coulomb barrier

For large sources (many thermal wavelengths),
— approximations #2, #3 and #4 should be good
— difficult to estimate accuracy for *pp* or e^+e^- collisions

Validity can depend on source of correlation (identical particles/strong/Coulomb)

Three Classes of Interaction

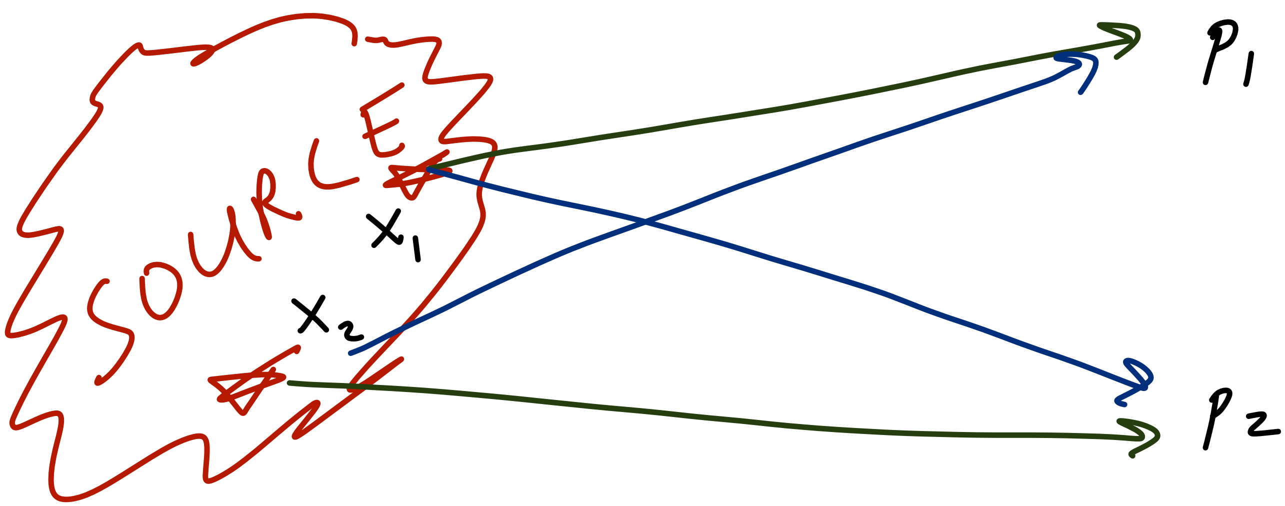
Identical Particle Statistics

$$U(x_1, x_2; \vec{p}_1, \vec{p}_2) = \frac{1}{\sqrt{2}}\left[e^{-ip_1\cdot x_1 - ip_2\cdot x_2} \pm e^{-ip_1\cdot x_2 - ip_2\cdot x_1}\right],$$

$$|U(x_1, x_2; \vec{p}_1, \vec{p}_2)|^2 = 1 \pm \cos\{(p_1 - p_2)\cdot(x_1 - x_2)\}$$

$$= 1 + \cos\{2\vec{q}' \cdot (\vec{x}_1' - \vec{x}_2')\} = |\phi_{\vec{q}}(\vec{r}')|^2$$

→2 as q→0



Three Classes of Interaction

Identical Particle Statistics / Gaussian Source

$$s(\vec{p},x) \sim \exp\left\{-\frac{x^2}{2R_x^2(\vec{p})} - \frac{y^2}{2R_y^2(\vec{p})} - \frac{z^2}{2R_z^2(\vec{p})} - \frac{t^2}{2\tau^2(\vec{p})}\right\},$$

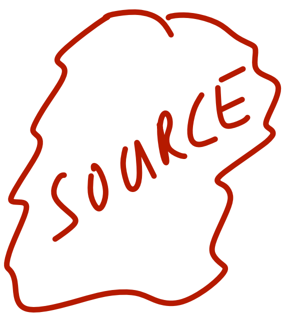
$$C(\vec{p},\vec{Q}) = 1 + \exp\left\{-Q_0^2\tau^2(\vec{p}) - Q_x^2R_x^2(\vec{p}) - Q_y^2R_y^2(\vec{p}) - Q_z^2R_z^2(\vec{p})\right\},$$

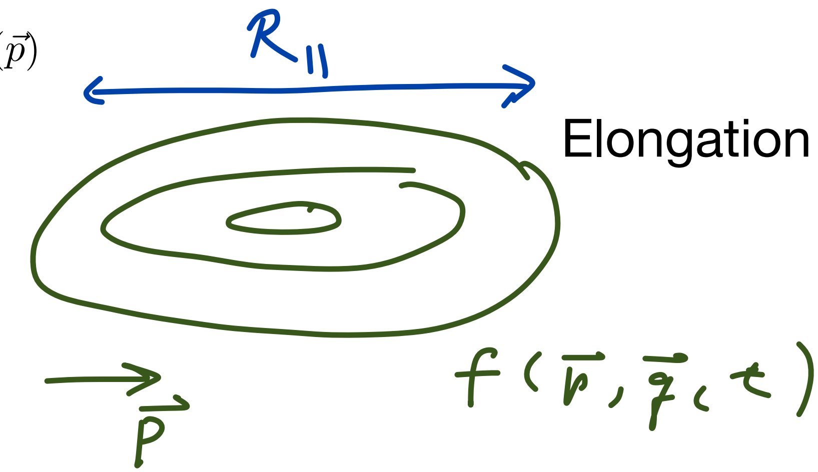
$$Q_0 = E_1 - E_2, \quad \vec{Q}_i = \vec{p}_1 - \vec{p}_2$$

Q_0 not independent

$$Q_0 \approx \vec{v}_p \cdot \vec{Q},$$

$$R_\parallel^2(\vec{p}) = (R_i(\vec{p}) \cdot \hat{v}_{p,i})^2 + v_p^2\tau^2(\vec{p})$$





Elongation along ***p*** signal of lifetime

Gaussian Source

Six parameters describe size for each *p*

$$S(\vec{r}) \sim \exp\left\{-\frac{1}{4}(R^2_{ij})^{-1} r_i r_j\right\}$$

$$C(\vec{p},\vec{Q}) = 1 + \exp\left\{Q_i R^2_{ij} Q_j\right\}$$

R^2 has six independent components

Equivalently: 3 radii and 3 Euler angles

Goal: determine 7-dimensional *s(p,r,t)*,
but measurement confined to 6-dimensional *C(p,Q)*
Temporal information always ambiguous

Gaussian Source

λ parameter (coherence parameter)

$$C(\vec{p},\vec{Q}) = 1 + \lambda \exp\left\{Q_i R_{ij}^2 Q_j\right\}$$

If fraction of interfering particles is $\lambda^{1/2}$

Fraction from long-lived resonances = 1- $\lambda^{1/2}$

Example: If 30% come from long-lived resonances, λ=0.49

Also used to describe effects of “coherence” — more later

Three Classes of Interaction

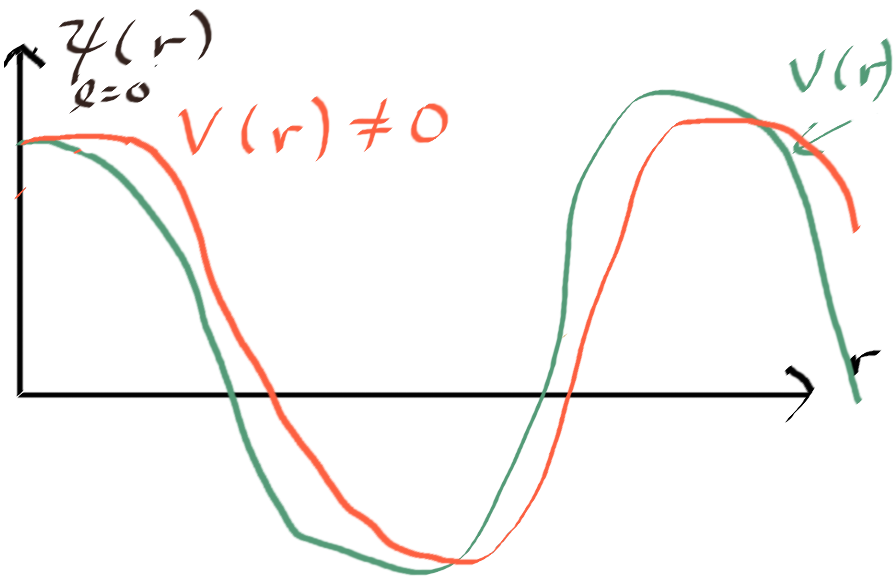
Strong Interaction

Most of source outside range of potential ⇒ only phase shifts matter

Usually only ℓ=0,1 are relevant

$$\text{For } r \gtrsim 1 \text{ fm}, \quad \phi_{\vec{q}}(\vec{r}) = e^{i\vec{q}\cdot\vec{r}} + \sum_{\ell}(2\ell+1)e^{i\delta_\ell}\sin\delta_\ell \frac{e^{iqr}}{qr}P_\ell(\cos\theta)$$

For r<1 fm, solve Schrödinger equation (not important for large sources)



Three Classes of Interaction

Strong Interaction / Density of States

For large volumes (qR>>1),

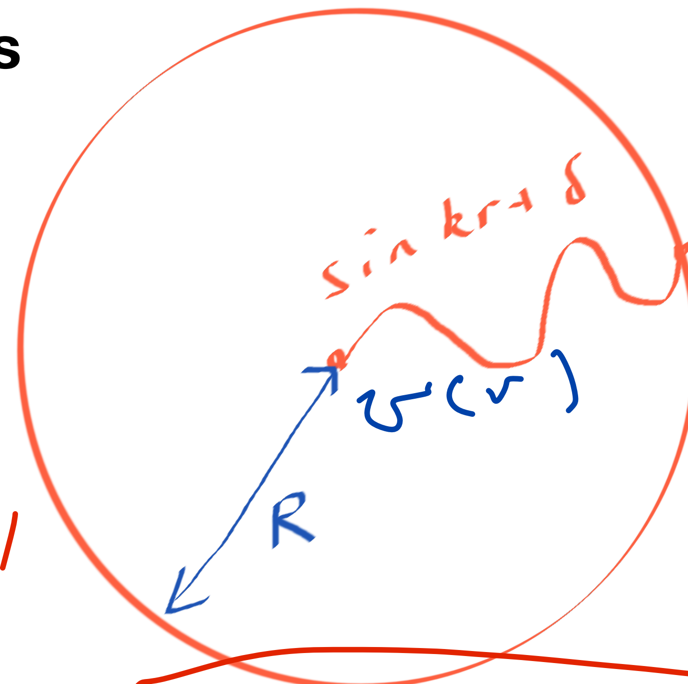
$$\sin(qR+\delta)=0,\quad qR+\delta=N\pi,$$

$$\frac{dN}{dq}=\left(\frac{dN}{dq}\right)_0+\frac{1}{\pi}\frac{d\delta}{dq},$$

$$C(q)=\frac{(2\ell+1)(1/\pi)d\delta_\ell/dq}{(4\pi q^2)V/(2\pi)^3}$$

$$=\frac{2\pi}{q^2V}\frac{d\delta}{dq}$$

$$\frac{1}{V}\rightarrow S(r=0)$$



Strong interaction good for measuring "volume"

Three Classes of Interaction

Strong Interaction / Large Volume Limit

For large volumes (qR>>1),

$$k_1^2 u_1 = -\partial_r^2 u_1 + 2m\hbar^2 V(r) u_1,$$ ← usual

$$k_2^2 u_2 = -\partial_r^2 u_2 + 2m\hbar^2 V(r) u_2,$$

$$k_1^2 w_1 = -\partial_r^2 w_1,$$ V=0

$$k_2^2 w_2 = -\partial_r^2 w_2.$$

For $r > 1$ fm, $w(r) = u(r) = \sin(kr + \delta)$

$$(k_1^2 - k_2^2)\int_0^R dr\ (u_1 u_2 - w_1 w_2)$$

$$= \int_0^R dr\ \left[-(\partial_r^2 u_1)u_2 + (\partial_r^2 u_2)u_1 - (\partial_r^2 w_1)w_2 + (\partial_r^2 w_2)w_1\right]$$

$$= \left[u_1\partial_r u_2 - u_2\partial_r u_1 + w_1\partial_r w_2 - w_2\partial_r w_1\right]_{r=0}$$

$$- \left[u_1\partial_r u_2 - u_2\partial_r u_1 + w_1\partial_r w_2 - w_2\partial_r w_1\right]_{r=R}$$

$$= k_2\sin(\delta_1)\cos(\delta_2) - k_1\cos(\delta_1)\sin(\delta_2)$$

Let $k_2 = k_1 + \Delta k$

$$2k\Delta k \int_0^{R\to\infty} dr\ \left(|\psi(r)|^2 - |\psi_0(r)|^2\right) = k\Delta\delta,$$

$$\int_0^{R\to\infty} dr\ \left(|\psi(r)|^2 - |\psi_0(r)|^2\right) = \frac{1}{2}\frac{d\delta}{dk}$$

Three Classes of Interaction

Strong Interaction / Large Volume Limit

$$\int_0^{R\to\infty} dr\ \left(|\psi(r)|^2 - |\psi_0(r)|^2\right) = \frac{1}{2}\frac{d\delta}{dk}$$

In terms of full scattering wave,

$$\int d^3r\left(|\phi_{\vec{q}}(\vec{r})|^2 - |\phi^{(0)}_{\vec{q}}(\vec{r})|^2\right) = \frac{2\pi}{q^2}\frac{d\delta}{dq},$$

$$\begin{aligned} C(q) &= 1 + \frac{\int d^3r\left(|\phi_{\vec{q}}(\vec{r})|^2 - |\phi^{(0)}_{\vec{q}}(\vec{r})|^2\right)}{\int d^3r} \\ &= 1 + \frac{2\pi}{q^2V}\frac{d\delta}{dq} \end{aligned}$$

Same as previous result!

Three Classes of Interaction

Strong Interaction / qR<<1, R>> 1 fm

$$qr << 1$$

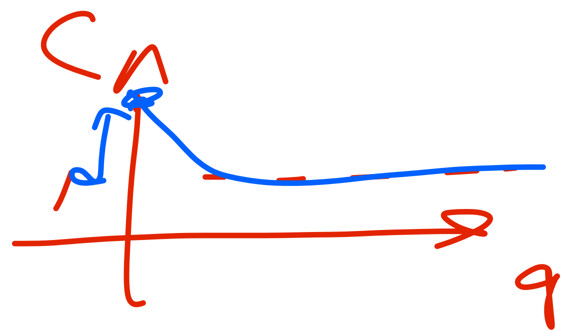
$$\delta \approx -qa$$

$$\phi_{\vec{q}}(\vec{r}) = e^{i\vec{q}\cdot\vec{r}} + e^{i\delta}\sin\delta\frac{e^{iqr}}{qr}$$

$$\approx 1 - \frac{a}{r},$$

$$|\phi_{\vec{q}}(\vec{r})|^2 \approx 1 - \frac{2a}{r},$$

$$C(q \to 0) \approx 1 - \int d^3r\ S(\vec{r})\frac{2a}{r}$$



C(q→0) determined by scattering length
For *nn* scattering length ~20 fm
Not useful when Coulomb present

Three Classes of Interaction

Coulomb Interaction

Classical Limit: Trajectory $\vec{q}_f(\vec{q}_i, \vec{r}_i)$

$$|\phi_{\vec{q}_f}(\vec{r}_i)|^2 \to \frac{d^3q_i}{d^3q_f}$$

Solve for q_i in terms of q_f

$$E = \text{constant} = \frac{q_f^2}{2m} = \frac{q_i^2}{2m} + \frac{e^2}{r_i},$$

$$L_y = \text{constant} = q_i r_i \sin\theta_i$$

$$\vec{q} \times \vec{L} + me^2\hat{r} = \text{constant} = r_i q_i^2(\sin\theta_i \hat{z} - \cos\theta_i \hat{x}) + me^2\hat{z}$$

$$= r_i q_i q_f(\cos\theta_f \hat{z} + \sin\theta_f \hat{x}) \times (\sin\theta_0 \hat{y}) + me^2(\cos\theta_f \hat{z} + \sin\theta_f \hat{x})$$

Lenz vector

$$q_i = \sqrt{q_f^2 - 2me^2/r_i}$$

$$\sin\theta_i = \frac{q_f \sin\theta_f \pm \sqrt{q_f^2 \sin^2\theta_f - 2(q_f^2 - q_i^2)(1 - \cos\theta_f)}}{2q_i}$$

$$\frac{d^3q_i}{d^3q_f} = \frac{q_i \sin\theta_i}{q_f \sin\theta_f} \frac{\partial}{\partial\theta_f}\theta_i$$

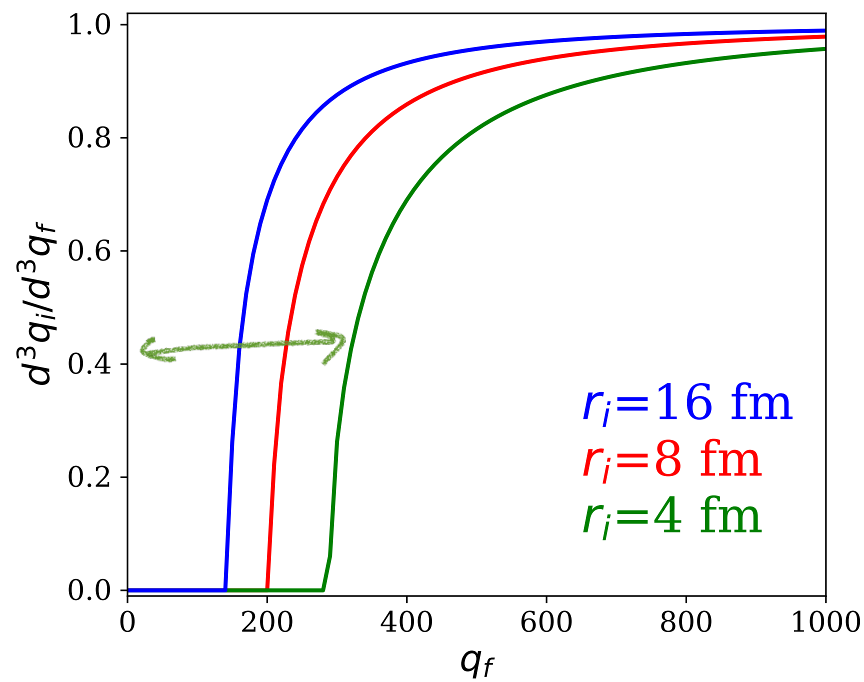
Good when $r/a_0 \gg 1$

Three Classes of Interaction

Coulomb Interaction

Classical Limit: Trajectory $\vec{q}_f(\vec{q}_i, \vec{r}_i)$

$$|\phi_{\vec{q}_f}(\vec{r}_i)|^2 \to \frac{d^3 q_i}{d^3 q_f}$$



Averaged over direction

Three Classes of Interaction
Coulomb Interaction

For $e^2/r >> q^2/2m$, tunneling

$$|\phi_{\vec{q}}(\vec{r} \to 0)|^2 = \frac{2\pi\eta}{e^{2\pi\eta} - 1},$$

$$\eta = \frac{me^2}{\hbar q}$$

Gamow penetration factor
finite probability of getting to origin

If $2\pi R/a_0 << 1$

$$|\phi_{\vec{q}}(\vec{r} \to 0)|^2 = \frac{2\pi\eta}{e^{2\pi\eta} - 1}[1 + \cos(2q \cdot r/\hbar)]$$

Gamow correction misses full Coulomb by ~10% for $\pi\pi$
Worse for larger sources or heavier particles

Three Classes of Interaction
Coulomb Interaction

Coulomb "Correction" often done by experiments

$$C_{\rm corrected}(\vec{q}) \rightarrow C_{\rm true}(\vec{q})/G(\eta),$$
$$G(\eta) = \frac{2\pi\eta}{e^{2\pi\eta}-1}$$

More sophisticated:

$$C_{\rm corrected}(\vec{q}) = C_{\rm true}(\vec{q})\frac{C^{(\rm Gaussian,\ noCoulomb)}(\vec{q})}{C^{(\rm Gaussian,\ withCoulomb)}(\vec{q})}$$

Gaussian requires choice of *R* and λ

Only done for **ππ**

Only purpose: satisfy lazy theorists

Three Classes of Interaction

Identical-Particle Interference

— Easy to invert
— Measures size and shape
— Excellent approximation for small q

Strong Interaction

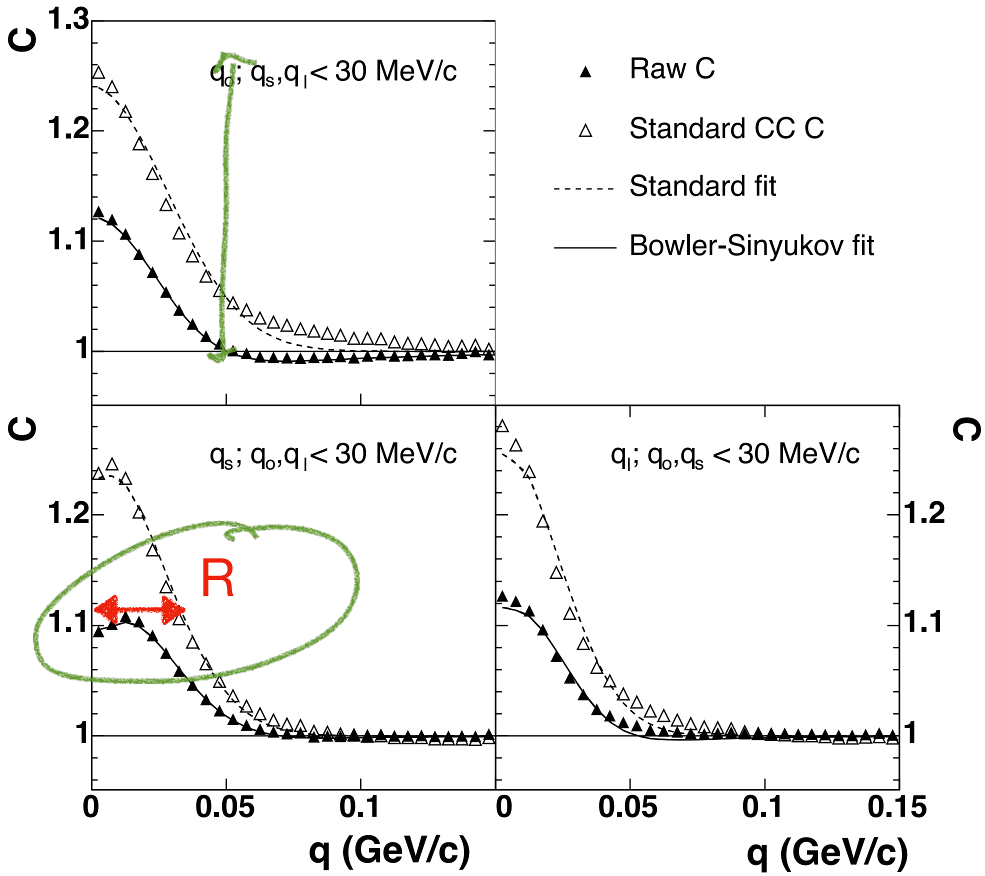
— Excellent for size, especially $S(r=0)$, less sensitive to shape
— Some theoretical "systematic error"

Coulomb Interaction

— Both size and shape for smaller Bohr radius (heavier or more highly charged)
— Theoretically robust if q is small
— For $\pi\pi$, impairs ability to use identical-particle interference

Three Classes of Interaction

Examples: Identical-Particle Interference

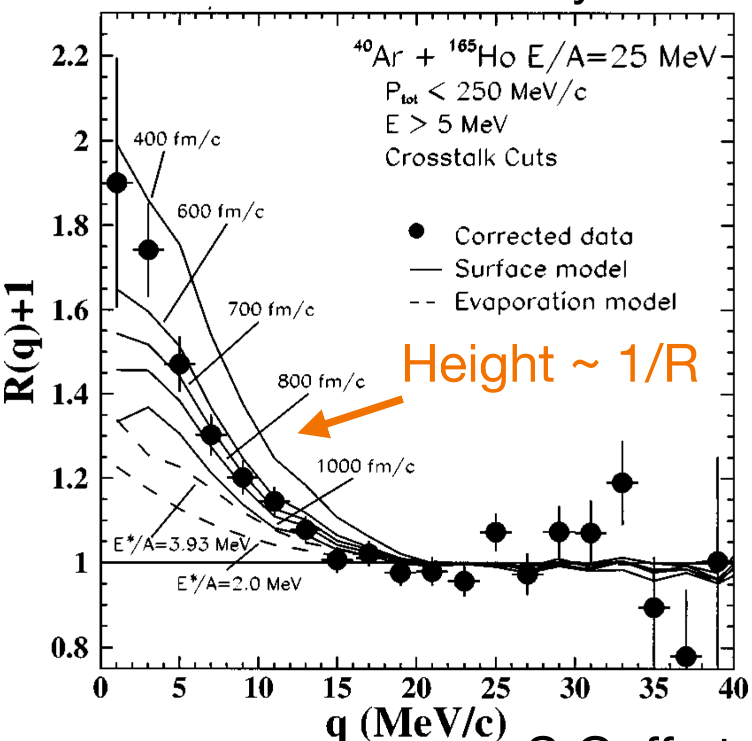


ππ interferometry
with & without Coulomb corrections

Three Classes of Interaction

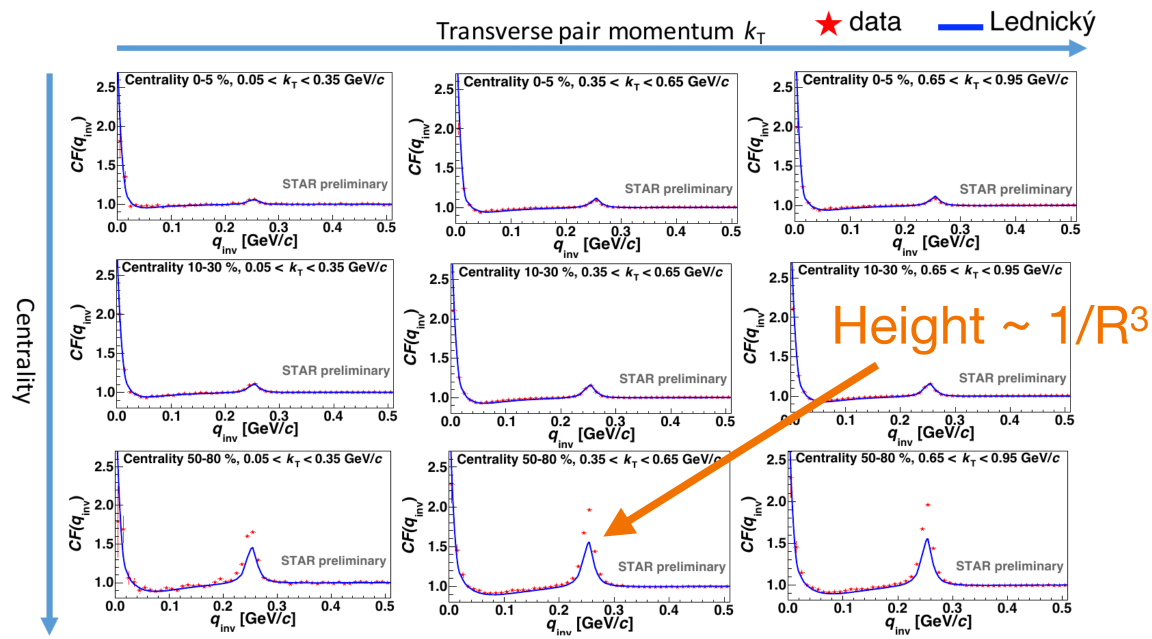
Examples: Strong Interaction

nn interferometry



S.Gaff et al, PRC 1998

K+K- correlations — φ peak



K. Mikhaylov, ALICE, WPCF 2019

Multi-Particle Symmetrization

Wave function has N! Terms

$$\phi(k_1, k_2, \cdots x_N; x_1, x_2, \cdots x_n) = \frac{1}{\sqrt{N!}} \sum_{\text{perm.}j(i)} \prod e^{\sum ik_i x_{j(i)}}$$

$$= \frac{1}{\sqrt{6}} \left(e^{ik_1x_1+ik_2x_2+ik_3x_3} + e^{ik_1x_2+ik_2x_3+ik_3x_1} + e^{ik_1x_3+ik_2x_1+ik_3x_2}\right.$$

$$\left. + e^{ik_1x_2+ik_2x_1+ik_3x_3} + e^{ik_1x_3+ik_2x_2+ik_3x_1} + e^{ik_1x_1+ik_2x_2+ik_3x_1}\right)$$

$$|\phi(k_1 \cdots k_n; x_1 \cdots x_n)|^2 = 1 + \cos[(k_1 - k_2) \cdot (x_1 - x_2)] + \cos[(k_2 - k_3) \cdot (x_2 - x_3)] + \cos[(k_1 - k_3) \cdot (x_1 - x_3)]$$

$$+ \cos[k_1 \cdot (x_1 - x_2) + k_2 \cdot (x_2 - x_3) + k_3 \cdot (x_3 - x_1)] + \cos[k_1 \cdot (x_1 - x_3) + k_2 \cdot (x_2 - x_1$$

$$|\phi(k_1 \cdots k_n; x_1 \cdots x_n)|^2 = 1$$

$$+ \cos[(k_1 - k_2) \cdot (x_1 - x_2)] + \cos[(k_2 - k_3) \cdot (x_2 - x_3)] + \cos[(k_1 - k_3) \cdot (x_1 - x_3)$$

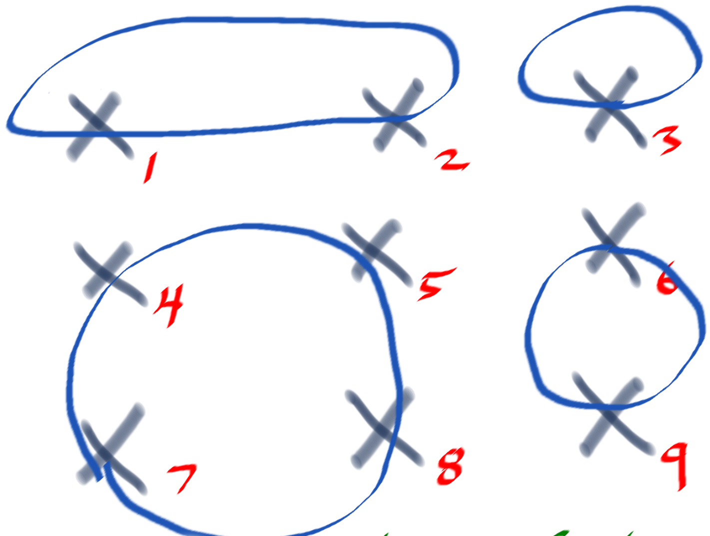
$$+ \cos[k_1 \cdot (x_1 - x_2) + k_2 \cdot (x_2 - x_3) + k_3 \cdot (x_3 - x_1)]$$

$$+ \cos[k_1 \cdot (x_1 - x_3) + k_2 \cdot (x_2 - x_1) + k_3 \cdot (x_3 - x_2)]$$

S.P., PLB 1993

Multi-Particle Symmetrization

Terms can be categorized via permutation cycles



$$G_3(p,q) = \int d^4x_1\, d^4x_2\, d^4x_3\, S\left(\frac{p+k_1}{2}, x_1\right) S\left(\frac{k_1+k_2}{2}, x_2\right) S\left(\frac{k_2+q}{2}, x_3\right) e^{i(p-k_1)\cdot x_1} e^{i(k_1-k_2)\cdot x_2} e^{i(k_2-q)\cdot x_3}$$

$$\Leftarrow C_n = \frac{1}{n}\int d^3p\, G_n(p,p)$$

$$w_N = \sum_n C_n w_{N-n}$$

↑ weights

Recursively sum over all permutations

$$P(p) = \frac{1}{w_N}\sum_n G_n(p,p)\, w_{N-n}$$

$$P(p,q) = \frac{1}{w_N}\sum_{n,m}\left[G_n(p,p)G_m(q,q) + G_n(p,q)G_m(q,p)\right]\cdot w_{N-n-m}$$

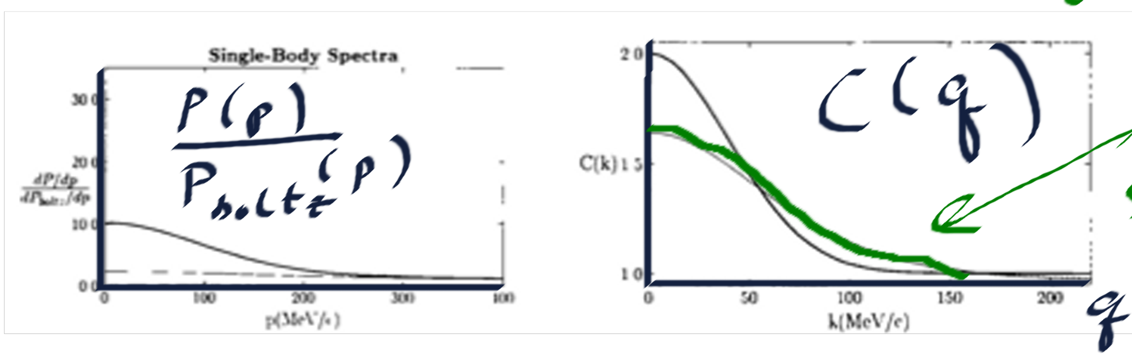
Analytical for non-relativistic Gaussian sources

S.P., PLB 1993

Multi-Particle Symmetrization

Multiplicity distributions, spectra and correlations distorted

At HIGH phase-space density



with symm.

Only important for pions at low p, where phase space density might be high
Adding pions to fixed phase space can result in super-luminescence

Multi-Particle Symmetrization

Coherent Sources

$$|\eta\rangle = \exp\left\{ i \int d^3p \; j(\vec{p})[a(\vec{p}) + a^\dagger(\vec{p})] \right\}$$

Results in no correlation,

$$\begin{aligned} \langle\eta|a^\dagger(\vec{p})a(\vec{p})|\eta\rangle &= j^*(\vec{p})j(\vec{p}), \\ \langle\eta|a^\dagger(\vec{p})a^\dagger(\vec{q})a(\vec{q})a(\vec{p})|\eta\rangle &= j^*(\vec{p})j(\vec{p})j^*(\vec{q})j(\vec{q}), \\ &= \langle\eta|a^\dagger(\vec{p})a(\vec{p})|\eta\rangle\langle\eta|a^\dagger(\vec{q})a(\vec{q})|\eta\rangle \end{aligned}$$

$$C(p,q) = 1$$

Some lasers described by coherent states,
is why λ is sometimes called “coherence parameter”
Many variations…

Multi-Particle Symmetrization

SUMMARY

— Unimportant unless phase space density is high

— Only an issue for $\pi\pi$ at low p_t ($\lessapprox$ 200 MeV/*c*)

— Dramatic behavior requires $\mu_\pi \sim m_\pi$ — unlikely from observations

— Difficult to predict manifestations once it becomes important, correlation can flatten (fall to 1.0) or broaden (maintain intercept at 2.0) depends sensitively on model assumptions

PHENOMENOLOGY — COLLECTIVE FLOW

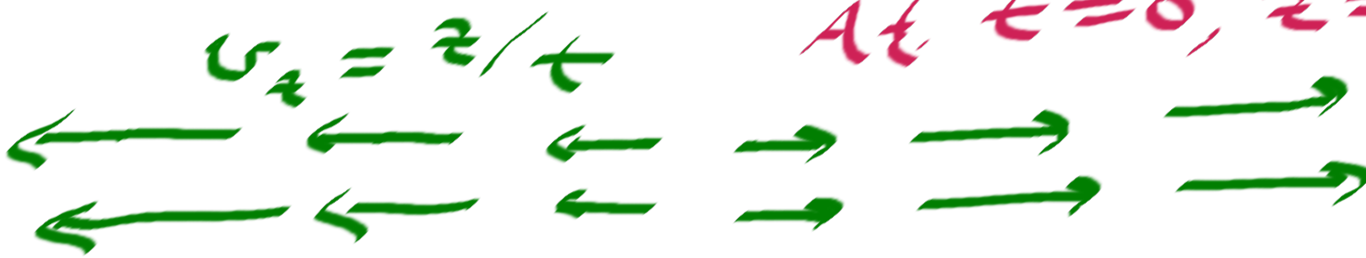
VOCABULARY: R_{out}, R_{long}, R_{side}

Only makes sense for HE collisions, Bjorken flow

$$v_{\mathrm{coll},z} = \frac{z}{t}, \quad \gamma v_{\mathrm{coll},z} = \frac{z}{\tau}, \quad y_{\mathrm{coll}} = \sinh(\eta),$$

$$\tau = \sqrt{t^2 - z^2} = \frac{t}{\gamma}, \quad \eta_s = \frac{1}{2}\ln\frac{1+z/t}{1-z/t} = \sinh^{-1}(\gamma v_{\mathrm{coll},z})$$

Colliding crepes
At $t=0$, $z=0$



— Boost-invariant: physics depends only on $\boldsymbol{\tau}$, not $\boldsymbol{\eta}_s$

— No longitudinal acceleration (coasting)

— Pair with rapidity y emitted mainly from matter moving at position $\boldsymbol{\eta}=y$

PHENOMENOLOGY — COLLECTIVE FLOW

VOCABULARY: R_{out}, R_{long}, R_{side}

Measure phase space cloud in LCMS (longitudinally comoving frame)

Relative momenta in LCMS

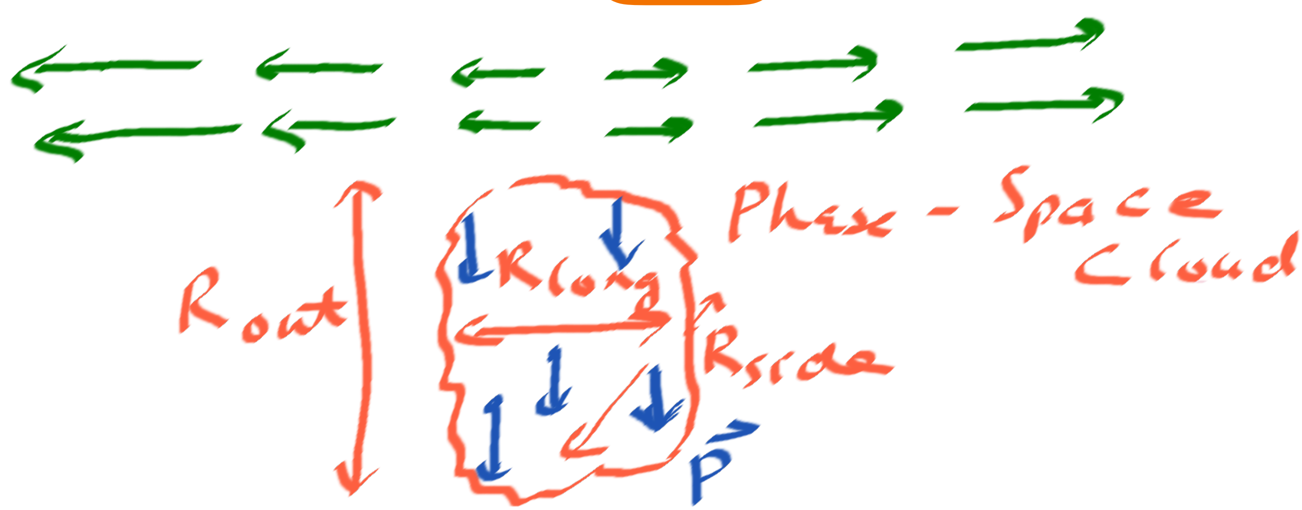
q_{long}	Longitudinal, along beam
q_{side}	Sideward, ⊥ to beam and ***p***
q_{out}	Outward, along ***p*** in LCMS

R_{long}

R_{side}

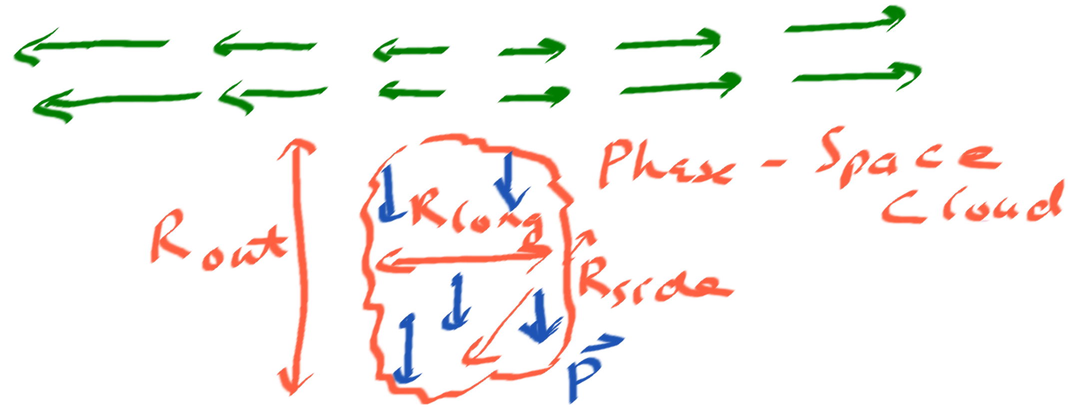
R_{out}

Gaussian dimensions in LCMS



PHENOMENOLOGY — COLLECTIVE FLOW

“Region of homogeniety”



Even if source is infinite, R_{long} is finite

$$R_{\text{long}} \sim v_{\text{therm}} \frac{dv_{\text{coll}}}{dz} = v_{\text{therm}} \tau_{\text{breakup}}$$

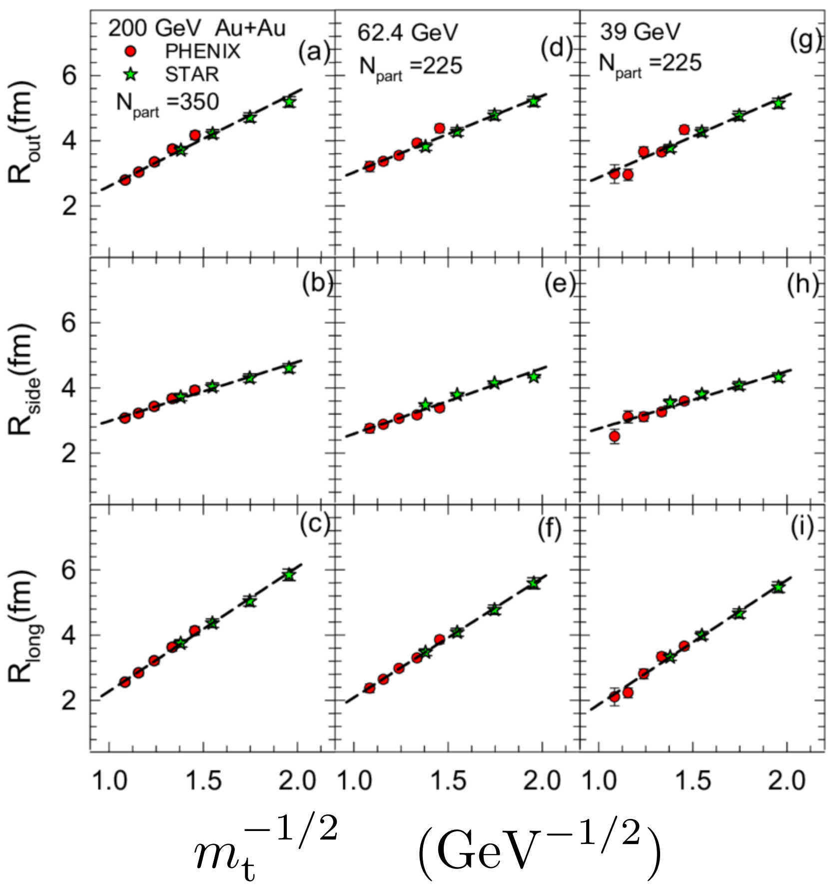
$$v_{\text{therm}} \sim \sqrt{\frac{T}{m_{\text{t}}}}, \quad m_{\text{t}} = \sqrt{m^2 + p_t^2}$$

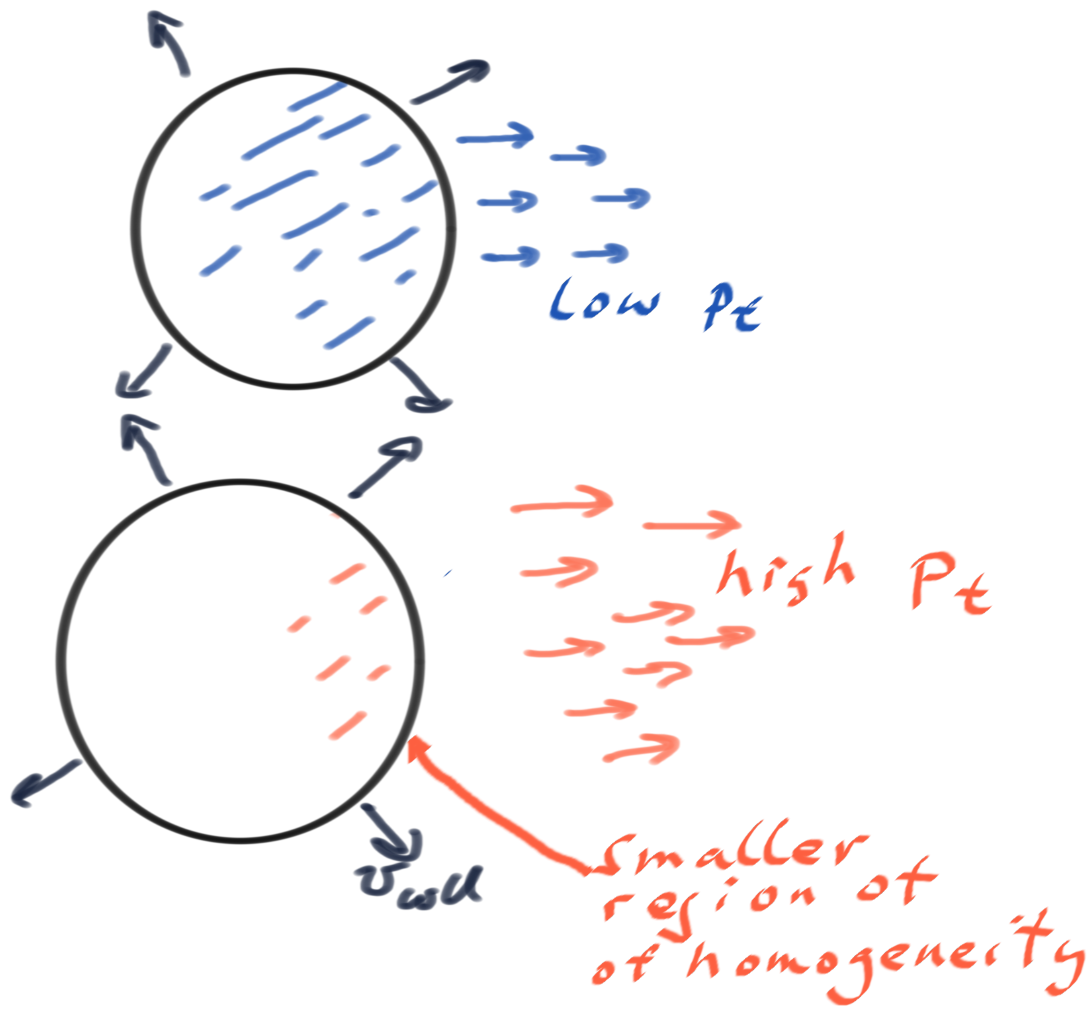
$$R_{\text{long}} \propto \frac{1}{\sqrt{m_{\text{t}}}}$$

M_{t} scaling for larger m_{t}
Independent of species
Only for R_{long}

PHENOMENOLOGY — COLLECTIVE FLOW

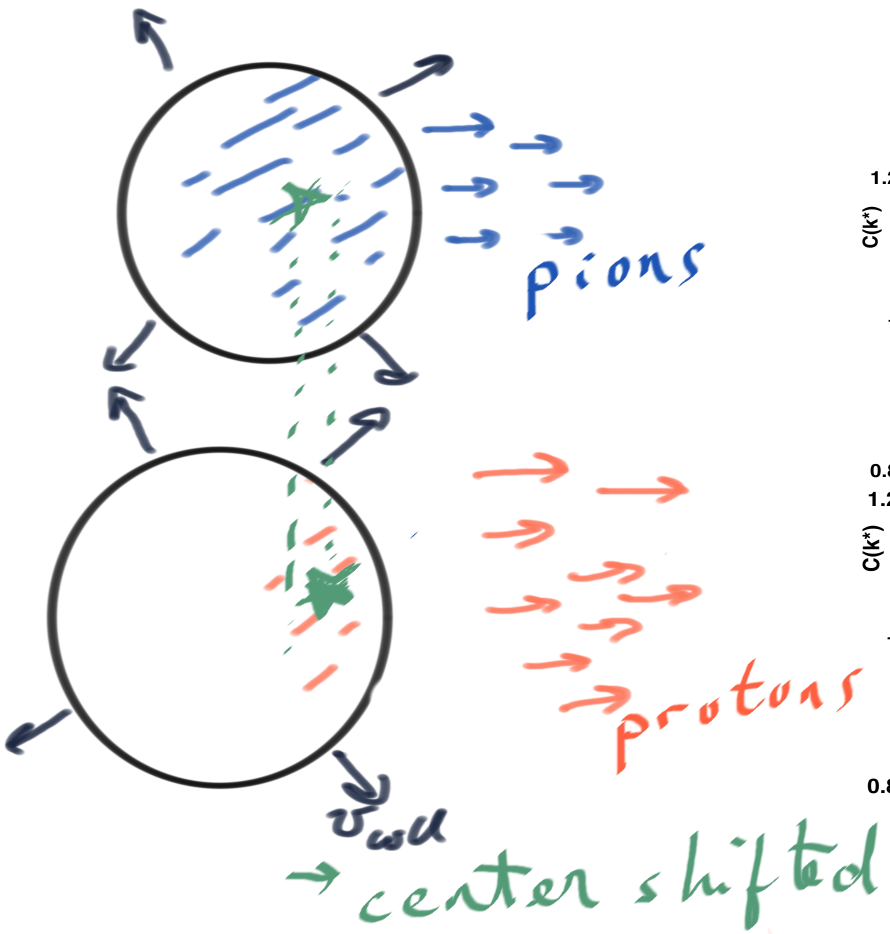
Transverse Flow



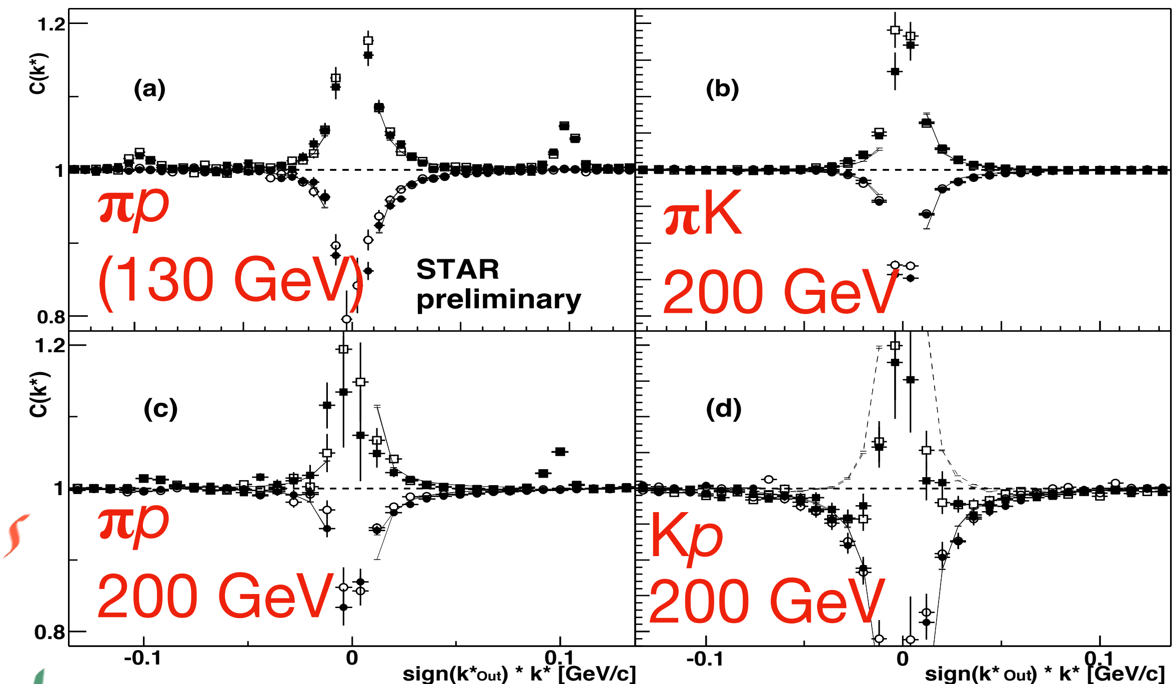


PHENOMENOLOGY — COLLECTIVE FLOW

Non-identical particles



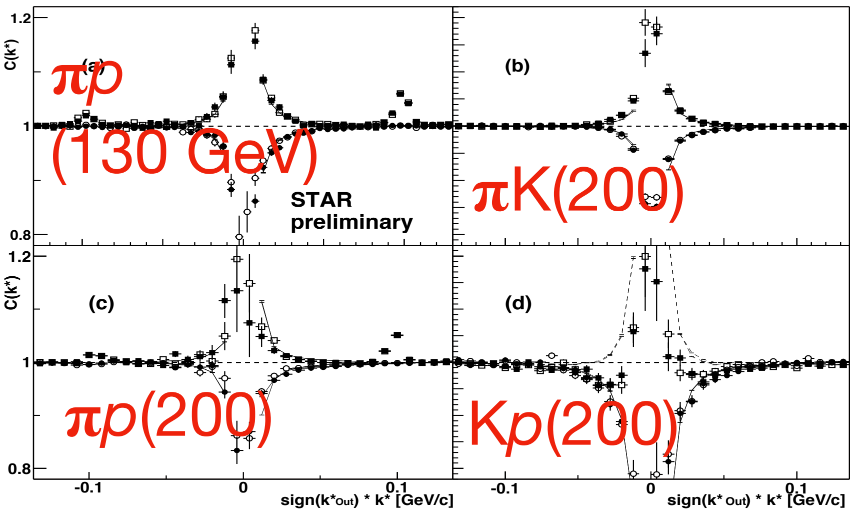
$$C(q_{\rm out}) \neq C(-q_{\rm out})$$



o -same sign
□ - opp. Sign

PHENOMENOLOGY — COLLECTIVE FLOW

Non-identical particles



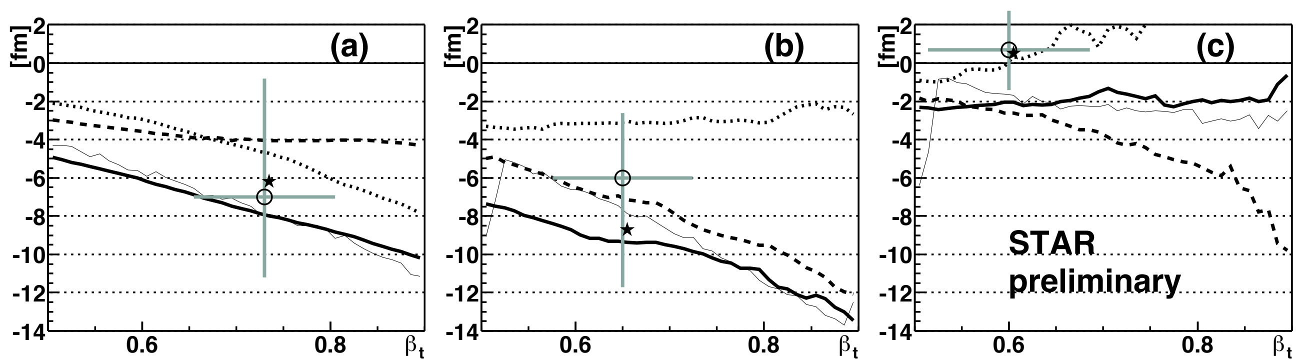
$$C(q_{\mathrm{out}}) \neq C(-q_{\mathrm{out}})$$

$$\langle z_a - z_b \rangle \neq 0$$

o -same sign

◻ - opp. Sign

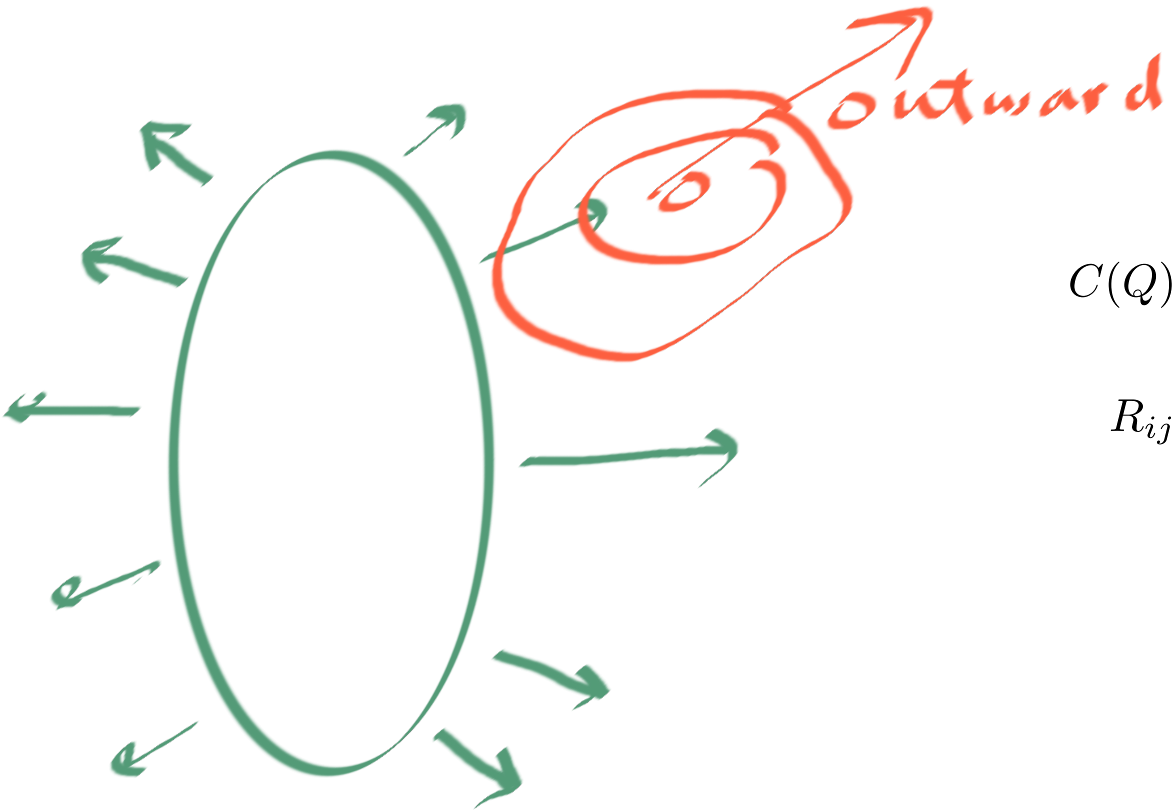
Offsets a few fm



PHENOMENOLOGY — COLLECTIVE FLOW

Elliptic Flow

out & side not necessarily principal axes



$$C(Q) \sim 1 + \exp\left(Q_i R_{ij}^2 Q_j\right)$$

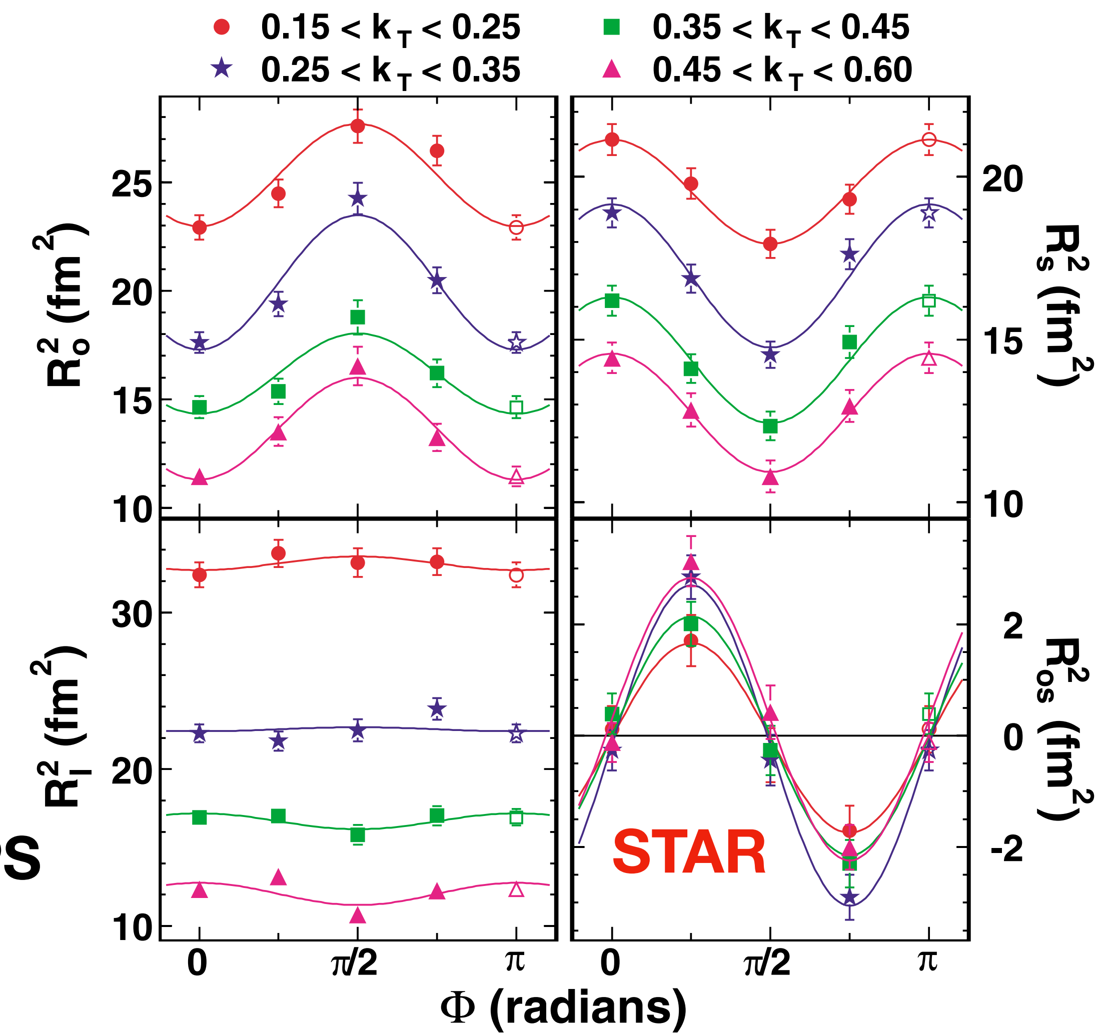
$$R_{ij} = \begin{pmatrix} R^2_{\text{out}-\text{out}} & R_{\text{out}-\text{side}} & R^2_{\text{out}-\text{long}} \\ R^2_{\text{side}-\text{out}} & R^2_{\text{side}-\text{side}} & R^2_{\text{side}-\text{long}} \\ R^2_{\text{long}-\text{out}} & R^2_{\text{long}-\text{side}} & R^2_{\text{long}-\text{long}} \end{pmatrix}$$

PHENOMENOLOGY — COLLECTIVE FLOW

Elliptic Flow

$$C(Q) \sim 1 + \exp\left(Q_i R_{ij}^2 Q_j\right)$$

$$R_{ij} = \begin{pmatrix} R^2_{\text{out-out}} & R_{\text{out-side}} & R^2_{\text{out-long}} \\ R^2_{\text{side-out}} & R^2_{\text{side-side}} & R^2_{\text{side-long}} \\ R^2_{\text{long-out}} & R^2_{\text{long-side}} & R^2_{\text{long-long}} \end{pmatrix}$$



***R^2*$_{\text{out-long}}$ & *R^2*$_{\text{side-long}}$ non-zero at SPS**

— violation of Bjorken flow

PHENOMENOLOGY — COLLECTIVE FLOW

Six dimensions of information

$$C(Y, p_t, \phi, Q_{\text{out}}, Q_{\text{side}}, Q_{\text{long}})$$

Relative momenta

Average momenta

Relative position

$$S(Y, p_t, \phi; r_{\text{out}}, r_{\text{side}}, r_{\text{long}})$$

Collective flow manifest in every dimension
All six dimensions have been explored!

Correlations from Transport Models

Algorithm(s)

First: Generate source function

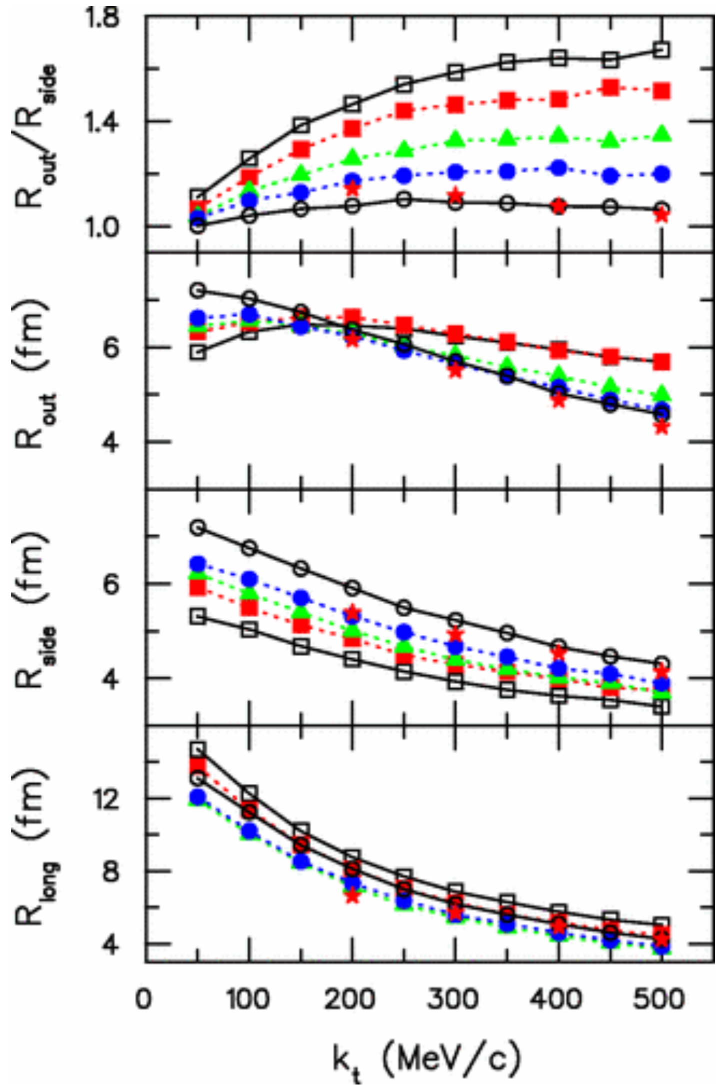
$$S_{ab}(\vec{v},\vec{r}) = \int d^3r_a d^3r_b \; \delta(\vec{r}_a - \vec{r}_b - \vec{r}) f_a(\vec{v},\vec{r}_a,t) f_b(\vec{v},\vec{r},t)$$

—Sum over every particle of type *a* and type *b* in velocity bin *v*
— Use positions and times of last interactions with remainder of system
—Make distribution of r_a-r_b-$v(t_a$-$t_b)$ [boost to c.o.m. frame]
—Average $|\boldsymbol{\varphi}(Q,r)|^2$ over pairs
— Carefully model acceptance
— Some variants of this procedure account for other correlations

Correlations from Transport Models

HBT Puzzle

R_{out}/R_{side} ~ 1.5 in models vs 1.0 in data



Solution:

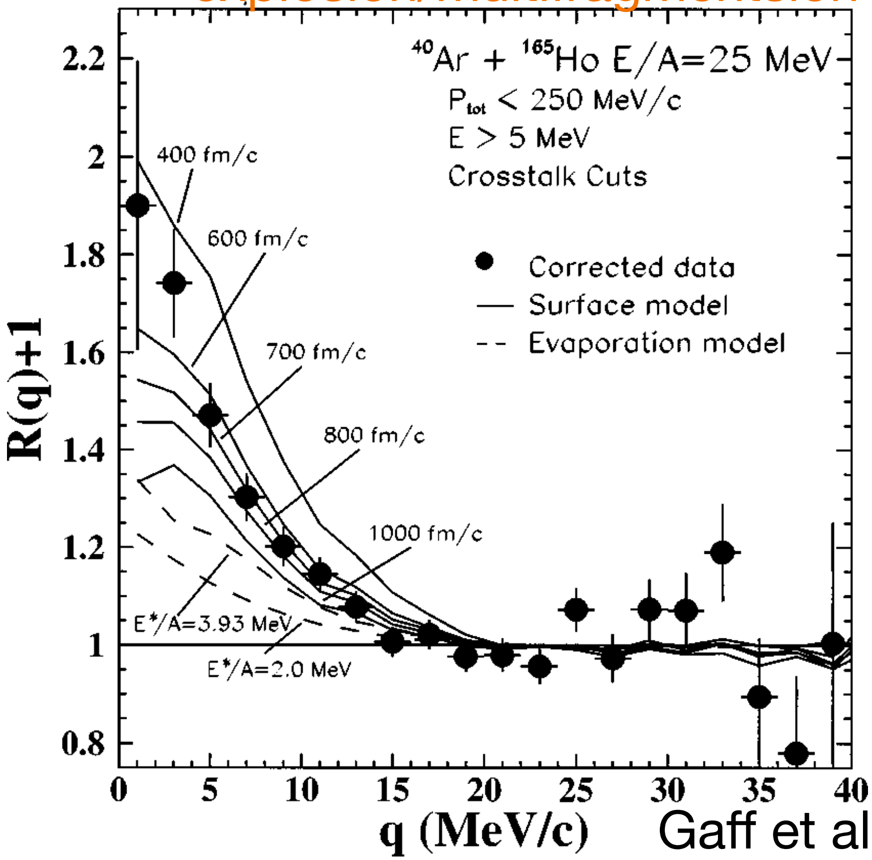
- **Include early collective flow**
- **Use reasonable equation of state**
- **Include viscosity**
- **Better relative wave function**

Be careful!

Extracting Eq. of State

Neutron-neutron correlations

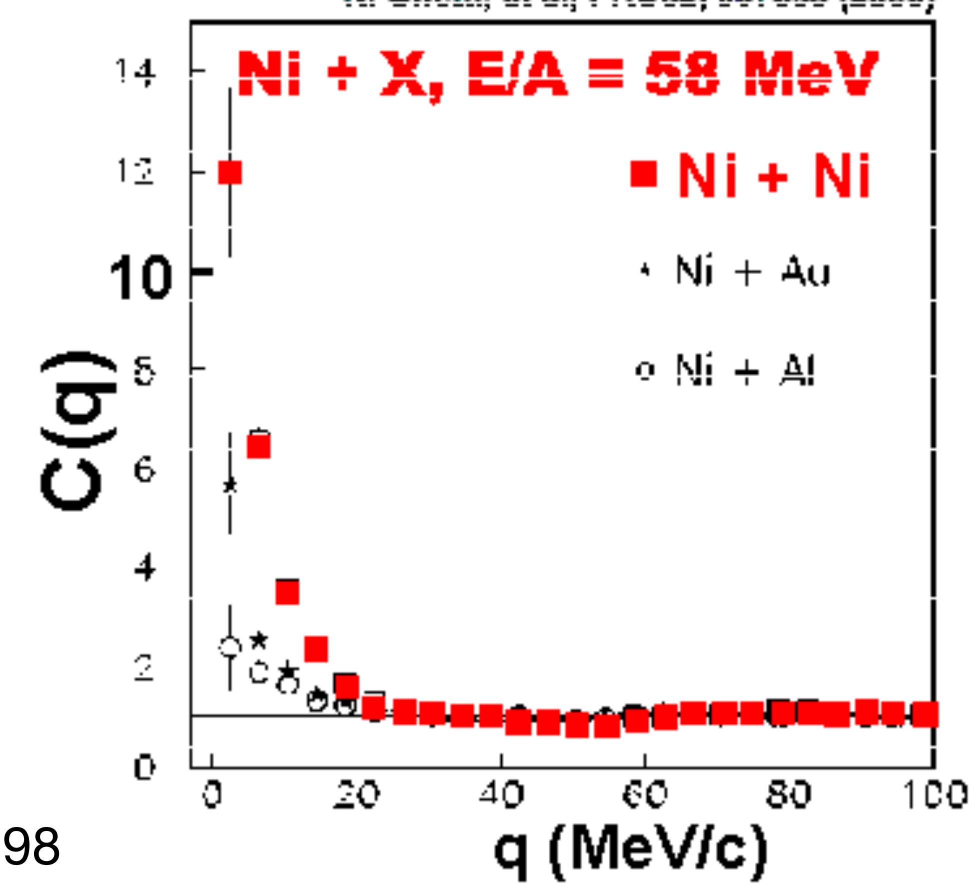
Below the energy for explosion/multifragmentsion



Gaff et al, PRC 98

Above the energy for explosion/multifragmentsion

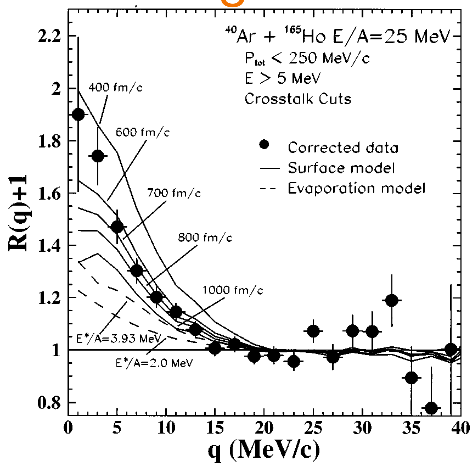
R. Ghetti, et al, PRC62, 037603 (2000)



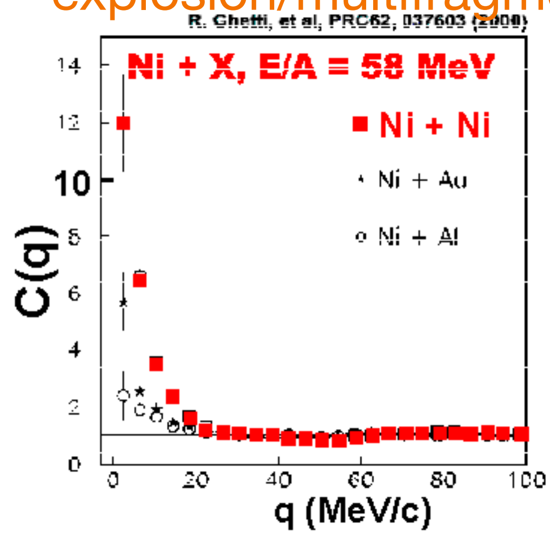
Extracting Eq. of State

Neutron-neutron correlations

Below the energy for explosion/multifragmentsion



Above the theshold for explosion/multifragmentsion

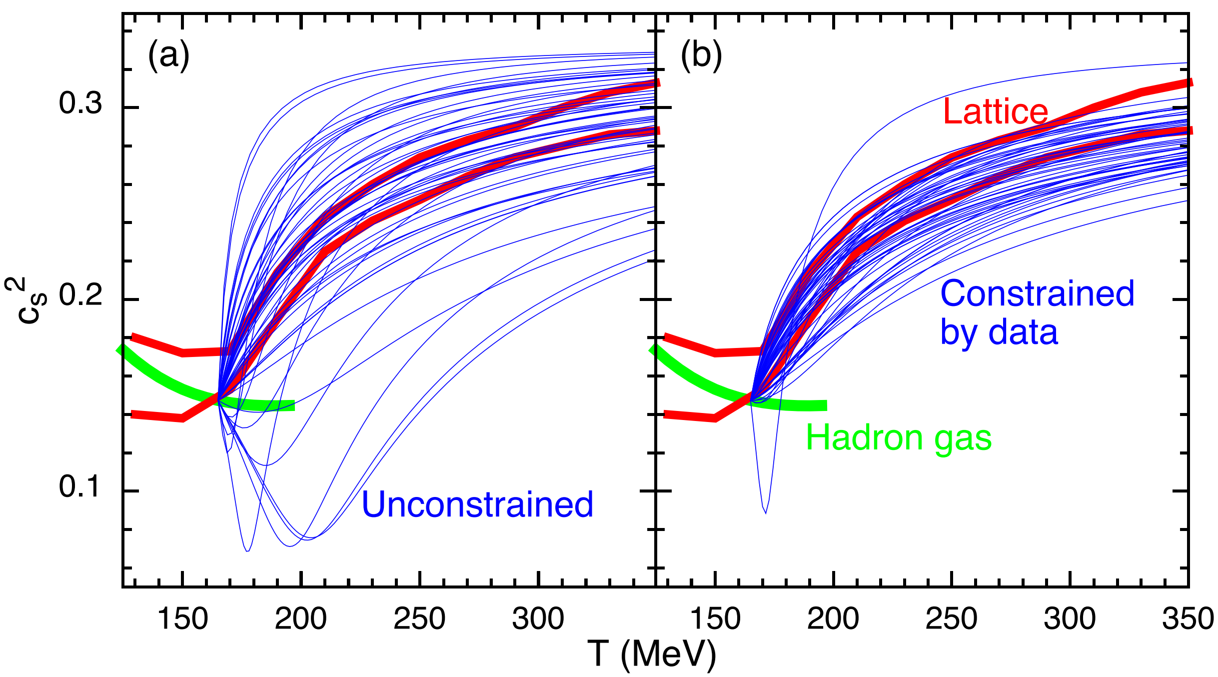


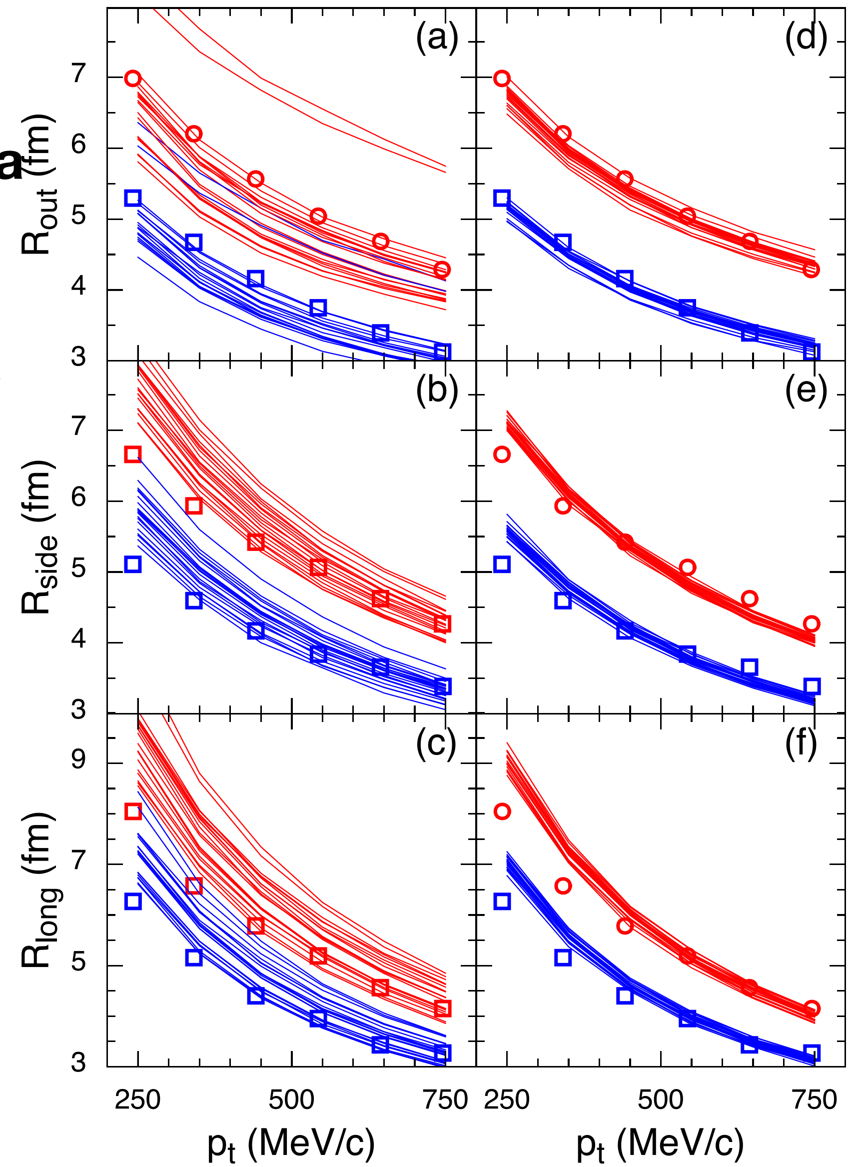
LIQUID: stays at finite density, cools ***slowly*** by evaporation
GAS: expands to fill all available volume, cools ***rapidly*** by expansion

Extracting Eq. of State

Bayesian analysis of LHC & RHIC (200 GeV) data

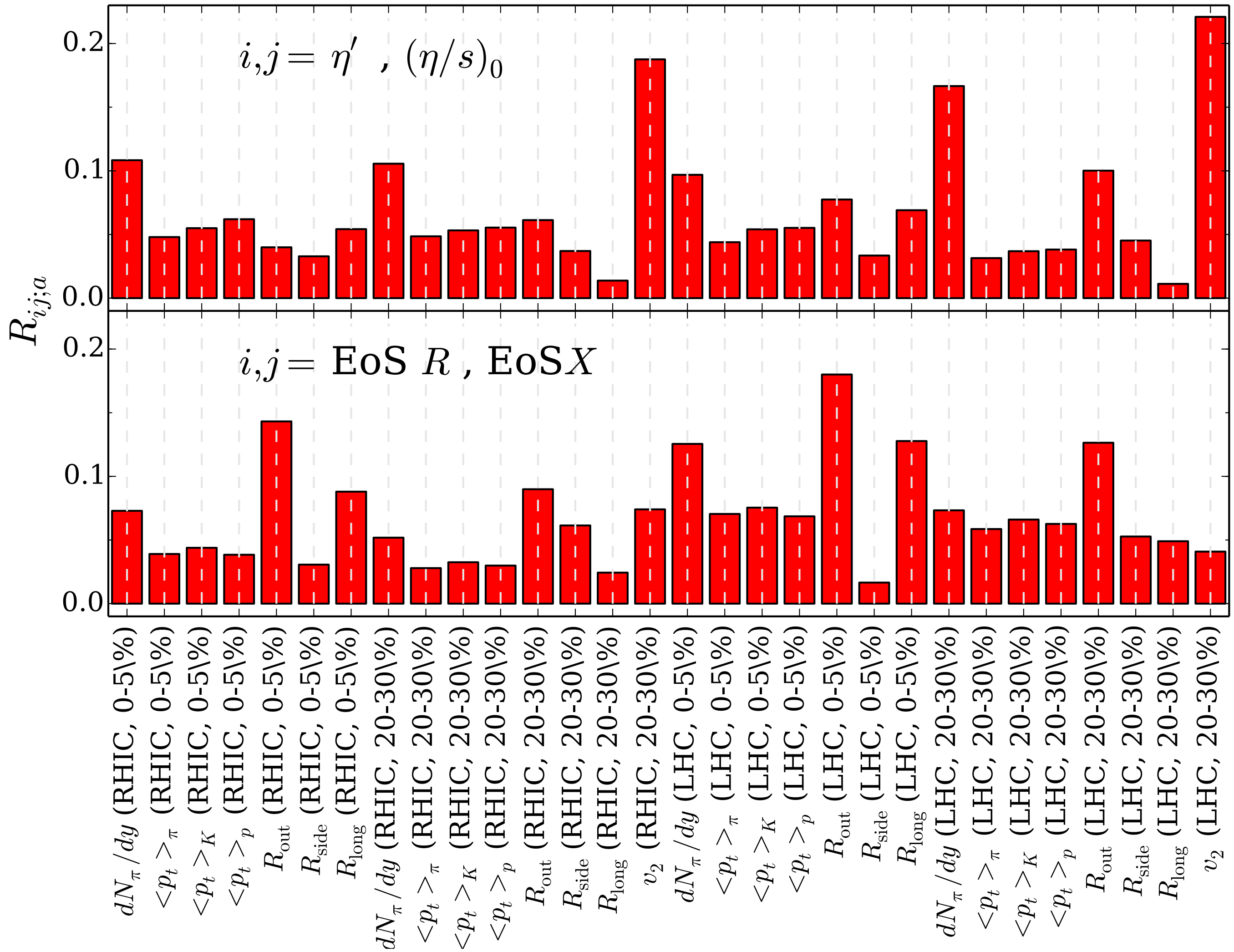
14-parameter analysis narrows down Eq. of State





Extracting Eq. of State

Bayesian analysis



Femtoscopic radii provide majority of resolving power

Imaging

The ultimate in femtoscopy!

D.Brown and P.Danielewicz, PRC 2001

$$C(\vec{q}) = \int d^3r \; |\phi_{\vec{q}}(\vec{r})|^2 S(\vec{r})$$

$|\boldsymbol{\varphi}|^2$ is function of q,r and $\cos\boldsymbol{\theta}$

Goal: invert C(q) to get S(r)
Strategy: break problem into spherical harmonics

Because $|\boldsymbol{\varphi}|^2$ is rotationally invariant (***q*** and ***r*** rotate together)

$C_{\ell m}(q)$ can only depend on $S_{\ell m}(r)$

Imaging

The ultimate in femtoscopy!

D.Brown and P.Danielewicz, PRC (2001)

$$C(\vec{q})-1=\sqrt{4\pi}\sum_{\ell_q m_q} C_{\ell_q m_q}(q)Y_{\ell_q,m_q}(\theta_q\phi_q),$$

$$C_{\ell_q m_q}(q)=\frac{\sqrt{4\pi}}{2\ell_q+1}\int d\cos\theta_q d\phi_q\; Y_{\ell_q m_q}(\theta_q,m_q)\left[C(\vec{q})-1\right]$$

$$S(\vec{r})=\sqrt{4\pi}\sum_{\ell_r m_r} S_{\ell_r m_r}(r)Y_{\ell_r m_r}(\theta_r,\phi_r),$$

$$S_{\ell_r m_r}(r)=\frac{\sqrt{4\pi}}{2\ell_r+1}\int d\cos\theta_r d\phi_r\; Y_{\ell_r m_r}(\theta_r,m_r)S(\vec{r})$$

$$K_L(q,r)=\frac{(2L+1)}{2}\int d\cos\theta_{qr}\; P_L(\cos\theta_{qr})\left[|\phi(q,r,\cos\theta_{qr})|^2-1\right],$$

$$|\phi(q,r,\cos\theta_{qr})|^2-1=\sum_L K_L(q,r)P_L(\cos\theta_{qr}),$$

$$\sum_{\ell m} C_{\ell m}(q)Y_{\ell m}(\theta_q,\phi_q)=\int r^2 dr\int d\cos\theta_r d\phi_r\;\left[|\phi(q,r,\cos\theta_{qr})|^2-1\right]\sum_{\ell_r m_r} S_{\ell_r m_r}(r)Y_{\ell_r m_r}(\theta_r,\phi_r)$$

$$=\sum_L\int r^2 dr\int d\cos\theta_r d\phi_r\; K_L(q,r)P_L(\cos\theta_{qr})\;\sum_{\ell_r m_r} S_{\ell m}(r)Y_{\ell_r m_r}(\theta_r,\phi_r),$$

$$P_L(\cos\theta_{qr})=\frac{4\pi}{2L+1}\sum_{m'=-L}^{L} Y^*_{Lm'}(\theta_q,\phi_q)Y_{L,m'}(\theta_r,\phi_r),$$

$$C_{\ell m}(q)=4\pi\int r^2 dr\; K_\ell(q,r)S_{\ell m}(r)$$

Imaging

The ultimate in femtoscopy!

$$K_L(q,r) = \frac{(2L+1)}{2} \int d\cos\theta_{qr} \; P_L(\cos\theta_{qr}) \left[|\phi(q,r,\cos\theta_{qr})|^2 - 1 \right],$$

$$C_{\ell m}(q) = 4\pi \int r^2 dr \; K_\ell(q,r) S_{\ell m}(r)$$

Image one ℓ,*m* at a time!

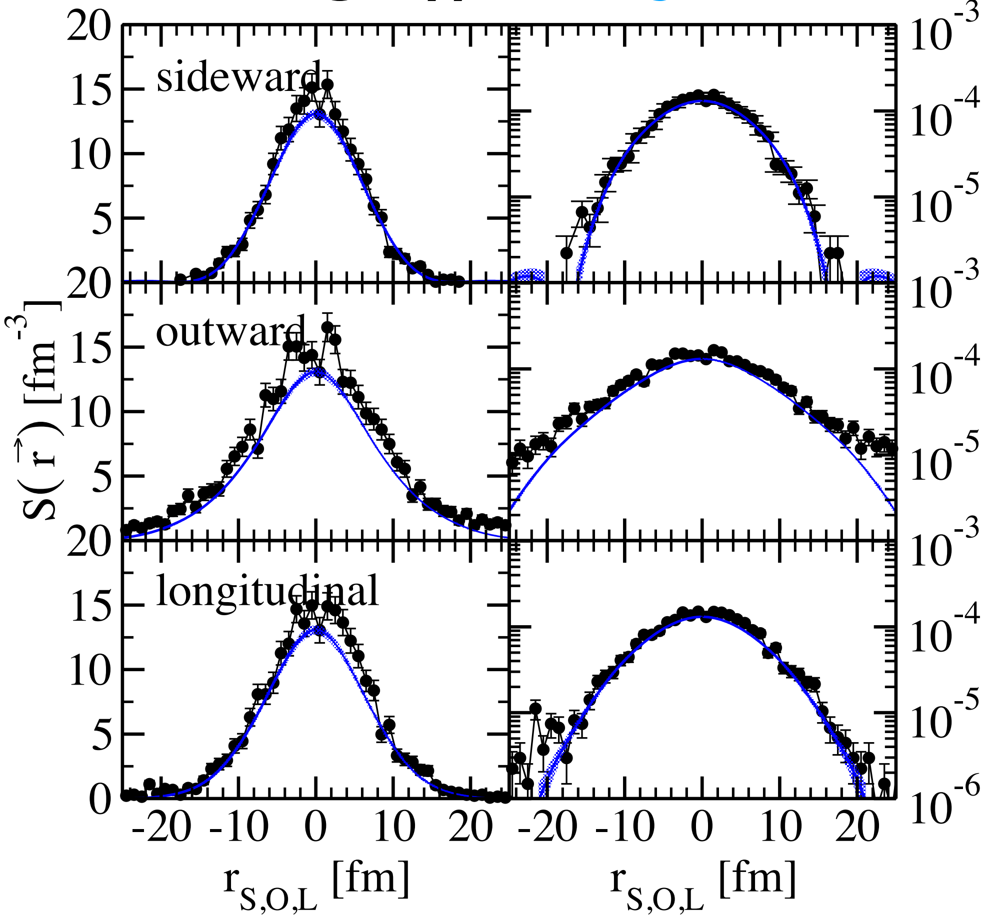
Kernel depends only on ℓ

Imaging

D.Brown,P.Danielewicz,M.Heffner & R. Soltz(2004)

consistency test

INPUT // IMAGED



from AGS data

GAUSSIAN // IMAGED