Planetoid

(a) $\vec{\nabla}\cdot\vec{g} = -4\pi G\rho$

$$\rho = -\frac{1}{4\pi G}\vec{\nabla}\cdot\vec{g} = -\frac{1}{4\pi G}\frac{1}{r^2}\frac{d}{dr}\left[\left(-\frac{g_0}{1-\alpha}\right)\frac{r^3}{R}\left(1-\alpha\frac{r}{R}\right)\right]$$

$$= \frac{1}{4\pi G}\frac{g_0}{1-\alpha}\frac{1}{R}\left[3-4\alpha\frac{r}{R}\right]$$

Since $\rho \geq 0$, we must have $3-4\alpha\frac{r}{R} \geq 0$

$3 \geq 4\alpha\frac{r}{R}$

worst situation for $r=R$

Conclusion $\alpha \leq \frac{3}{4}$.

(b) $$M = \int \rho\, dV = 4\pi\int_0^R dr\; r^2\; \frac{1}{4\pi G R}\frac{g_0}{1-\alpha}\left(3-4\alpha\frac{r}{R}\right)$$

$$= \frac{1}{GR}\frac{g_0}{1-\alpha}\int_0^R dr\left(3r^2-\frac{4\alpha}{R}r^3\right)$$

$$= \frac{1}{GR}\frac{g_0}{1-\alpha}\cdot\left[3\frac{r^3}{3}-\frac{4\alpha}{R}\frac{r^4}{4}\right]\Bigg|_0^R$$

$$= \frac{1}{G}\frac{g_0}{1-\alpha}R^2\left[1-\alpha\right] = \frac{g_0R^2}{G}$$

(c) $4\pi R^2 g(R) = -4\pi G M \quad \therefore \quad M = -\frac{R^2}{G}(-g_0) = \frac{g_0R^2}{G}$ OK.

Skater

(a) $\ell = r_i m v_i = 55\,kg \times 3.1\,\frac{m}{s} \times 3.4\,m$

$= 579.7\,kg\,m^2/s$

(b) $\ell = r_f m v_f$

$579.7 = 1.7\,m \times 55\,kg \times v_f$

$v_f = \frac{\ell}{1.7\,m \times 55\,kg} = 6.2\,\frac{m}{s}$

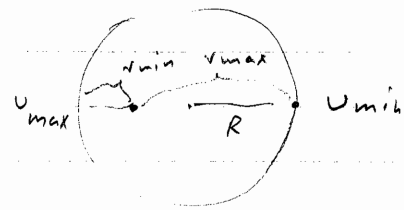
(c) $W = K_f - K_i = \frac{m v_f^2}{2} - \frac{m v_i^2}{2}$

$= \frac{55}{2}\left(6.2^2 - 3.1^2\right) = 793\,J$

(d) $\frac{\ell^2}{2m r_f^2} - \frac{\ell^2}{2m r_i^2} = \frac{m^{\cancel{2}} \cancel{r_f^2} v_f^2}{2 \cancel{m} \cancel{r_f^2}} - \frac{m^2 \cancel{r_i^2} v_i^2}{2m \cancel{r_i^2}}$

$= \frac{m v_f^2}{2} - \frac{m v_i^2}{2} = W$

Circular Orbit



$$\ell = v_{max}\, m\, r_{min} = v_{min}\, m\, r_{max}$$

$$\ell = m r^2 \dot{\theta} \quad \Rightarrow \quad \frac{\ell}{m} = r^2 \dot{\theta}$$

$$\text{const} = \frac{1}{2}\frac{\ell}{m} = \frac{1}{2} r^2 \dot{\theta} = \frac{dA}{dt} = \frac{A}{T}$$

$$T = \frac{2Am}{\ell} = \frac{2\pi R^2 m}{\ell}$$

$$\ell = m v_{max} r_{min} = m v_{min}(2R - r_{min})$$

$$v_{max} r_{min} = v_{min}(2R - r_{min})$$

$$\frac{v_{max}}{v_{min}} r_{min} = 2R - r_{min}$$

$$\left(1 + \frac{v_{max}}{v_{min}}\right) r_{min} = 2R \quad \Rightarrow \quad r_{min} = \frac{2 v_{min} R}{v_{min} + v_{max}}$$

$$\ell = m v_{max} r_{min} = m \frac{2 v_{max} v_{min} R}{v_{min} + v_{max}}$$

$$T = \frac{\cancel{2}\pi R^{\cancel{2}} \cancel{m}}{\cancel{m}\, \cancel{2} \frac{v_{max} v_{min} \cancel{R}}{v_{min} + v_{max}}} = \frac{\pi R (v_{min} + v_{max})}{v_{min} \cdot v_{max}}$$

Stationary satellite

(a) $T = 24h$

$$\frac{\cancel{m}v^2}{r} = \frac{\cancel{m}MG}{r^2} \qquad v = \frac{2\pi r}{T}$$

$$\frac{1}{r}\left(\frac{2\pi r}{T}\right)^2 = \frac{MG}{r^2}$$

$$\frac{4\pi^2 r}{T^2} = \frac{MG}{r^2}$$

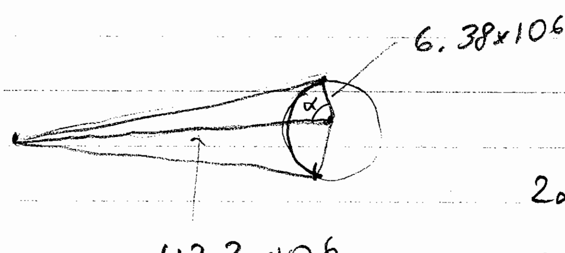
$$\frac{4\pi^2 r}{T^2} = \frac{gR^2}{r^2}$$

$$r^3 = g\frac{R^2T^2}{4\pi^2} = \left(\frac{RT}{2\pi}\right)^2 g$$

$$r = \sqrt[3]{\left(\frac{RT}{2\pi}\right)^2 g} = \sqrt[3]{9.81\,\frac{m}{s^2}\left[\frac{24\times3600\,s\times 6.38\times10^6\,m}{2\pi}\right]^2}$$

$r = 4.23\times10^7\ m$

(b)



$$\cos\alpha = \frac{R}{r} \quad \therefore \quad \alpha = \cos^{-1}\frac{6.38}{42.3} = 81.3$$

$2\alpha = 162.7°$

$$\frac{360°}{2\alpha} = 2.21 \Rightarrow \text{3 satellites needed}$$

Inverse-Cube Law

$$\frac{\cancel{m} v^2}{r} = F = G\frac{\cancel{m} M}{r^3}$$

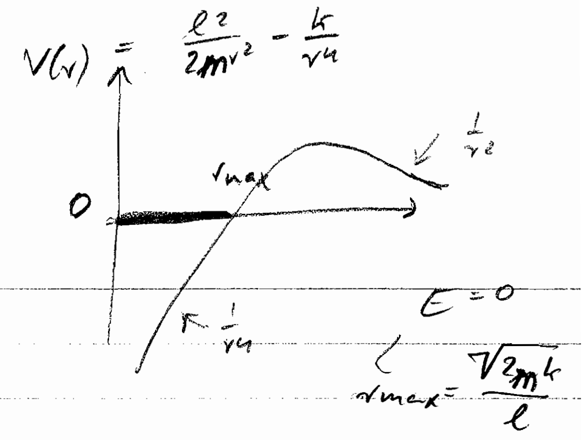
$$\frac{(2\pi r)^2}{r T^2} = G\frac{M}{r^3}$$

$$\frac{4\pi^2 r}{T^2} = G\frac{M}{r^3}$$

$$T^2 = \frac{4\pi^2 r^4}{GM}$$

$$T = \frac{2\pi r^2}{\sqrt{GM}} \qquad \text{i.e.} \qquad T \propto r^2$$

$1/r^4$ potential



$$\theta = \pm \int \frac{\ell}{r^2} \frac{dr}{\sqrt{2m\left(\frac{k}{r^4} - \frac{\ell^2}{2mr^2}\right)}}$$

$$= \pm \ell \int \frac{dr}{\sqrt{2mk - \ell^2 r^2}} = \pm \frac{\ell}{\sqrt{2mk}} \int \frac{dr}{\sqrt{1 - \frac{\ell^2 r^2}{2mk}}}$$

$$y = \frac{\ell r}{\sqrt{2mk}} \qquad = \pm \int \frac{dy}{\sqrt{1-y^2}}$$

$$y = \sin\alpha \qquad dy = \cos\alpha \, d\alpha$$

$$\sqrt{1-y^2} = \cos\alpha$$

$$\theta = \pm \alpha + \text{const}$$

$$\alpha = \sin^{-1} \frac{\ell r}{\sqrt{2mk}}$$

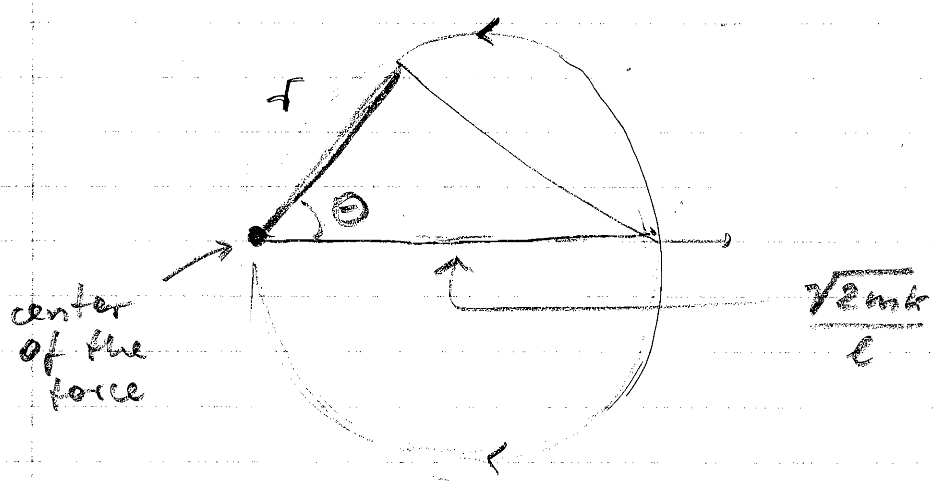
$$\theta = \pm \sin^{-1} \frac{\ell r}{\sqrt{2mk}} + \text{const}$$

$$\frac{\ell r}{\sqrt{2mk}} = \sin(\pm \theta + \text{const})$$

$1/r^4$ potential

$$r = \frac{\sqrt{2mk}}{l} \sin(\pm\theta + \text{const})$$

Range of θ, for which r may be maintained positive, is π.



For this situation, $\text{const} = \frac{\pi}{2}$

$$r = \sqrt{\frac{2mk}{l}} \cos\theta$$