Name: Solutions **Student ID:** ____________

S14 PHY321: Midterm 2

March 19, 2014

NOTE: Show all your work to maximize partial credit. *No credit* for unsupported answers.

Turn the front page only when advised by the instructor!

Check that your exam has all 3 problems. Total points: **15**

Formulas pertaining to the material:

Rocket equation: $v = -gt + u \log(m_0/m)$

Damped harmonic oscillator equation: $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$

General solution: $x(t) = \mathrm{e}^{-\beta t}\left[A_1 \exp\left(\sqrt{\beta^2 - \omega_0^2} t\right) + A_2 \exp\left(-\sqrt{\beta^2 - \omega_0^2} t\right)\right]$

Driven harmonic oscillator equation: $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = A\cos\omega t$

Amplitude of stationary driven oscillations: $D = \frac{f}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + 4\omega^2\beta^2}}$

Phase lag of driven oscillations: $\delta = \tan^{-1}\left(\frac{2\omega\beta}{\omega_0^2 - \omega^2}\right)$

Avg. power in driven oscillator: $0 = \langle P_{\text{drive}}\rangle + \langle P_{\text{damp}}\rangle$

1. A rocket is ascending against gravity. The rocket engine delivers a thrust of $\tau = 1.50 \times 10^5\,\mathrm{N}$.

   (a) [1 pt] At what rate is the fuel being burnt, if the speed of the exhaust gas is $u = 3050\,\mathrm{m/s}$?

   (b) [2 pts] What is the maximal mass m_0 that this rocket could have for the take-off to occur right away?

   (c) [2 pts] How long would it take for the rocket with the maximal mass m_0 to reach the mass ratio of $m_0/m = 3.50$?

a) $\tau = \alpha u \Rightarrow \alpha = \frac{1.5\times10^5\,\mathrm{N}}{3050\,\mathrm{m/s}} = \underline{49.18\,\frac{\mathrm{kg}}{\mathrm{s}}}$

b) $\tau \geq m_0 g \Rightarrow m_0^{max} = \frac{\tau}{g} = \frac{1.5\times10^5\,\mathrm{N}}{9.8\,\frac{\mathrm{m}}{\mathrm{s}^2}} = \underline{1.53\times10^4\,\mathrm{kg}}$

c) $\frac{dm}{dt} = -\alpha \Rightarrow m(t) = m_0 - \alpha t$

$\frac{m(t)}{m_0} = 1 - \frac{\alpha}{m_0}t$

$\Rightarrow t = \frac{m_0}{\alpha}\left(1 - \frac{m(t)}{m_0}\right) = \frac{1.53\times10^4}{49.18}\left(1 - \frac{1}{3.5}\right)$

$= \underline{222\,\mathrm{s}}$

2. A particle of mass m is moving along the x-axis under the influence of a conservative force for which the potential energy is given by

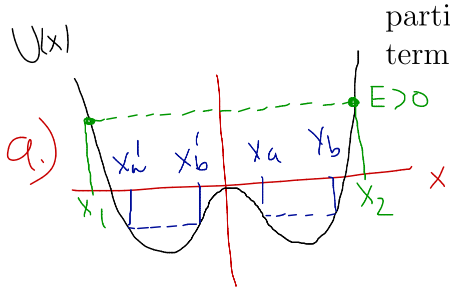
$$U(x) = -Bx^2 + Cx^4$$

where B, and C are positive constants.

(a) [2 pt] Sketch the form of the potential energy function and qualitatively describe the motion of the particle for i) $E > 0$ and ii) $U_{min} < E < 0$.

(b) [3 pts] Find the natural frequency ω_0 of small oscillations of the particle for motion near the minima x_0 of the potential energy, in terms of m, B, and C.

a.)



(i) $E>0$ particle oscillates between x_1 & x_2

(ii) $U_{min} < E < 0$ particle oscillates about one of the minima (between x'_a & x'_b for the left one, between x_a & x_b for the right minimum. Note no motion between x'_b & x_a is possible.)

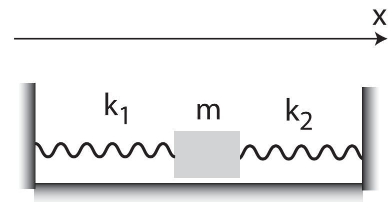
b.) Find minimum x_0: $\left.\frac{dU}{dx}\right|_{x_0} = 0 = -2Bx_0 + 4Cx_0^3 \Rightarrow x_0 = \pm\sqrt{\frac{B}{2C}}$

Near x_0, $U(x) \simeq \underbrace{\cancel{U}(x_0)}_{\text{constant, ignore}} + (x-x_0)\underbrace{\cancel{\left.\frac{dU}{dx}\right|_{x_0}}}_{0} + \frac{1}{2}(x-x_0)^2 U''(x_0) \Rightarrow U(x) \simeq \frac{1}{2}U''(x_0)(x-x_0)^2$

Now, $U_{HO} = \frac{k}{2}(x-x_0)^2 \Rightarrow k = U''(x_0) = -2B + 12Cx_0^2 \overset{\text{plug in}}{\underset{x_0}{=}} 4B$

$$\boxed{\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{4B}{m}}}$$

3. A small object of mass m is positioned on a smooth plane and attached to walls on its two sides with stretched massless horizontal springs of spring constants k_1 and k_2, as displayed in the figure.

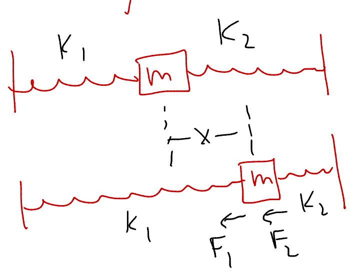


(a) [1 pts] What is the net force acting on the mass m when it is in its equilibrium position? What is the net force acting on the mass m when it is displaced by x from the equilibrium position, expanding by that distance one of the springs and compressing the other?

(b) [2 pts] What is the angular frequency of oscillations about the equilibrium position, in absence of friction? Compute the value for $m = 2.50\,\text{kg}$, $k_1 = 6.50\,\text{N/m}$ and $k_2 = 3.50\,\text{N/m}$.

(c) [2 pts] If a friction force is further applied on the mass, opposite and proportional to the velocity, $F_f = -b\,\dot{x}$, with a proportionality constant of $b = 0.80\,\text{kg/s}$, does the motion become underdamped, overdamped or critically damped?

a.) $F = 0$ @ equilibrium by definition. When displaced by X we have



equilibrium

displaced by X

$$\boxed{F = -K_1 X - K_2 X = -(K_1 + K_2) X}$$

b.) $\omega_0 = \sqrt{\dfrac{k_1 + k_2}{m}} = 2 \text{ rad/s}$

c.) $\beta = \dfrac{b}{2m} = \dfrac{.8}{2 \cdot 2.5} = .16\, s^{-1}$ since $\beta < \omega_0 \Rightarrow$ Underdamped.

Scratch Paper

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