Solution Key

PHY321 Homework Set 3

1. [5 pts] An object is ejected straight up into the air at an initial velocity v_0.

   (a) Determine the time for reaching the maximal elevation when the object is subject to gravity alone.

   (b) Determine the time for reaching the maximal elevation when the object is subject to gravity combined with a retarding force of the form $-kmv$.

   (c) Demonstrate that your result from 1b agrees with 1a in the limit of $k \to 0$. *Note:* It is not sufficient just to put $k = 0$ as this is likely to yield an ill-defined result. Rather carefully employ Taylor expansion in your result from 1b, assuming a small k, to demonstrate the agreement when $k \to 0$.

   (d) On the basis of your expansion from 1c decide whether the retarding force extends or shortens the time to reach the maximal elevation.

2. [5 pts] A woman of mass m sits on a train that coasts along the tracks at a constant speed u. She gets up and begins to walk forward along the car at the speed v relative to the car.

   (a) What is the gain in kinetic energy of the woman for a passenger sitting on the train?

   (b) What is the gain in kinetic energy of the woman for an observer standing on a station?

   (c) How much work has the woman done in order to put herself into motion relative to the car?

   (d) How much additional work was done on the woman by the car while she was putting herself into motion? Hint: To make sure that your answer is correct, take into account the action-reaction law and the impulse-momentum theorem. As the woman pushes with her feet against the floor of the car, the floor acts with a reaction force onto the feet. That force increases the momentum of the woman relative to the car. During the time when the force acts, the woman is displaced relative to an outside observer, because of the motion of the car.

   (e) If the woman started walking in a moving lightweight cart, instead of the car of a train, what might be the evidence of the work done by the cart, for an outside observer?

3. [5 pts] Determine the location of the center of mass of a uniform solid cone of base radius R and height H.

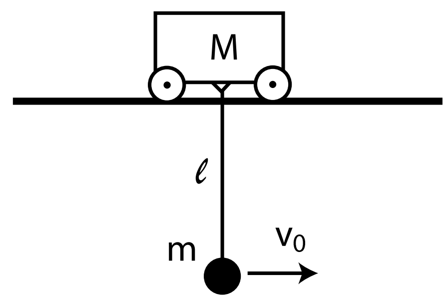
4. [5pts] In testing a missile defense system, a missile is fired from the ground on a trajectory that would directly hit a bunker some distance away. When the missile is at the top of the trajectory, a laser light from the bunker ignites fuel in the missile and the missile disintegrates into two pieces, one twice as massive as the other. The pieces reach the ground nearly simultaneously, 60 m apart from each other.

   (a) By how much does the larger piece miss the bunker? Hint: Consider motion of the center of mass.

   (b) By how much does the smaller piece miss the bunker?

   (c) How important is the information that the fuel ignited at the top of the trajectory?

5. [5 pts] Ball of mass m hangs on a string of length ℓ straight down from a cart of mass M standing on horizontal rails. The cart can move along the rails without friction. While the cart is at rest, the ball is given a horizontal velocity v_0, $v_0 < \sqrt{2g\ell}$, directed along the rails.



   (a) What is velocity of the ball when it reaches its maximal elevation?

   (b) What is elevation h that the ball reaches above its original location?

1.) Particle thrown up in the air

a.) No air resistance $\Rightarrow \cancel{m}\frac{dv}{dt} = -\cancel{m}g \Rightarrow v(t) = v_0 - gt$

* at top $v(t) = 0 \Rightarrow \boxed{t = \frac{v_0}{g}}$

b.) Air resistance $F_{damp} = -kmv \Rightarrow \cancel{m}\frac{dv}{dt} = -\cancel{m}g - \cancel{m}kv$

$$\int -\frac{dv}{g+kv} = \int dt$$

$$-\frac{1}{k}\ln(g+kv)\Big|_{v_0}^{0} = t$$

$$\boxed{\therefore t = -\frac{1}{k}\ln g + \frac{1}{k}\ln(g+kv_0) = \frac{1}{k}\ln\left(1+\frac{kv_0}{g}\right)}$$

c.) to take the $k \to 0$ limit, use the Taylor series

$$\ln\left(1+\frac{kv_0}{g}\right) \simeq \left(\frac{kv_0}{g}\right) - \frac{1}{2}\left(\frac{kv_0}{g}\right)^2 + \mathcal{O}(k^3) \quad \text{valid for } k \to 0$$

$$\boxed{\therefore \lim_{k\to 0}\left[\frac{1}{k}\ln\left(1+\frac{kv_0}{g}\right)\right] = \frac{1}{k}\left(\frac{kv_0}{g}\right) = \frac{v_0}{g} \quad \text{as it must}}$$

d.) Looking at the next terms in c) for small (but finite) k:

$$t \simeq \frac{1}{k}\left[\frac{kv_0}{g} - \frac{1}{2}\left(\frac{kv_0}{g}\right)^2 \cdots\right] = t_{\text{no-resistance}} - \frac{1}{2}k\frac{v_0^2}{g^2}$$

$\therefore$ t is smaller w/ air resistance.

2) Woman on a train w/ constant speed u

a) $\boxed{\Delta T_{woman} = \frac{1}{2} m v^2}$ (m = woman's mass, v = her speed w.r.t. the train)

b) $\Delta T = \frac{1}{2} m (v+u)^2 - \frac{1}{2} m u^2$

$$= \frac{1}{2} m v^2 + m v u + \cancel{\frac{1}{2} m u^2} - \cancel{\frac{1}{2} m u^2}$$

$\therefore$ As seen from the station, $\boxed{\Delta T_{woman} = \frac{1}{2} m v^2 + m v u}$

c) $W_{woman} = \Delta T = \frac{1}{2} m v^2$

d) The change in momentum (the impulse) from the woman's feet on the floor is

$$F_{feet} \Delta t = \Delta P = (m(u+v) + Mu) - (mu + Mu) = mv$$

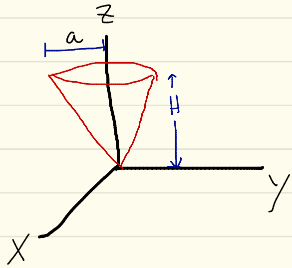
Now, by newton's 3rd law $\vec{F}_{train} = -\vec{F}_{feet}$, and

$$W_{train} = F_{train} \Delta x = F_{train} u \Delta t = F_{feet} u \Delta t = \frac{mv}{\cancel{\Delta t}} \cdot u \cancel{\Delta t}$$

$$\boxed{\Rightarrow W_{train} = m v u}$$

e) The cart slows down, losing kinetic energy.

3) CM of a uniform cone



$$\vec{R} = \frac{\int \rho \vec{r}\, d^3x}{\int \rho\, d^3x} = \frac{\int \vec{r}\, d^3x}{\int d^3x} = X\hat{\imath} + Y\hat{\jmath} + Z\hat{k}$$

since $\rho(\vec{r}) = \rho$ (const)

Symmetry $\Rightarrow$ $\vec{R} = Z\hat{k}$

$$\Rightarrow Z = \frac{\int z\, d^3x}{\int d^3x}$$

$$d^3x = (\pi r^2(z))\, dz$$

$$r(z) = \left(\frac{z}{H}\right)a$$

$$\boxed{\therefore Z = \frac{\int_0^H z^3\, dz}{\int_0^H z^2\, dz} = \frac{\frac{1}{4}H^4}{\frac{1}{3}H^3} = \frac{3}{4}H}$$

4.) Disintegrated Projectile

a.) Recall that the CM moves under the influence of uniform gravity as if the total system's mass concentrated at $\vec{R}$. The disintegration involves no external forces, so the motion of the CM is not impacted. I.e, the CM passes thru the targeted bunker.

Taking the bunker as the reference point, we have

$$0 = m_{small} r_{small} - m_{large} r_{large} \qquad \text{where } d = r_{small} + r_{large}$$

$$\Rightarrow 0 = m_{small} d - r_{large}(m_{small} + m_{large})$$

$$\boxed{\Rightarrow r_{large} = \frac{d\, m_{small}}{m_{small} + m_{large}} = \frac{d}{1 + \frac{m_{large}}{m_{small}}} = \frac{d}{3} = 20\,m}$$

b.) $\boxed{\begin{aligned} r_{small} &= d - r_{large} \\ &= 40\,m \end{aligned}}$

c.) It doesn't matter at what point it disintegrates.

5.) Cart w/ball

a) When the ball is at its highest point, its velocity <u>relative to the cart</u> vanishes. Therefore, Cons. of Momentum gives

$$m v_0 = (m + M_{cart}) v$$

$$\Rightarrow \boxed{v = \frac{m}{m+M} v_0}$$

b) Energy Cons. $\Rightarrow$ $\frac{1}{2} m v_0^2 = \frac{1}{2}(m+M)v^2 + mgh$

* plug in v from a) $\Rightarrow$ $\frac{1}{2} m v_0^2 = \frac{1}{2} \frac{m^2}{(m+M)} v_0^2 + mgh$

* Solve for h $\Rightarrow$ $h = \frac{1}{mg} \cdot \frac{1}{2} m v_0^2 \left(1 - \frac{m}{m+M} \right)$

$$\boxed{h = \frac{1}{2g} v_0^2 \frac{M}{m+M}}$$