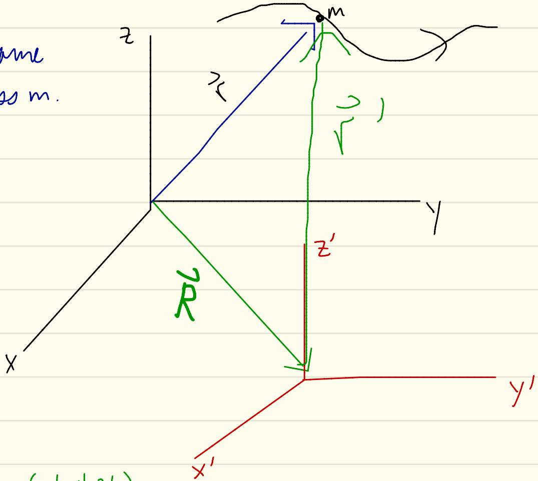
Motion in Non-inertial Reference Frames

* Consider the inertial frame to describe motion of mass m.



$$\vec{F} = m\vec{a} = m\frac{d^2\vec{r}}{dt^2}$$
$$\vec{r} = (x, y, z)$$
$$\vec{v} = (\dot{x}, \dot{y}, \dot{z})$$
$$\vec{a} = (\ddot{x}, \ddot{y}, \ddot{z})$$

* Consider the coordinates (x', y', z')

$$\vec{r} = \vec{R} + \vec{r}\,'$$
$$\dot{\vec{r}} = \dot{\vec{R}} + \dot{\vec{r}}\,'$$
$$\ddot{\vec{r}} = \ddot{\vec{R}} + \ddot{\vec{r}}\,'$$

Case 1: $\ddot{\vec{R}} = 0$ (primed frame inertial too) $\Rightarrow \ddot{\vec{r}} = \vec{a} = \vec{a}\,' = \ddot{\vec{r}}\,'$

$\Rightarrow$ Even though observer A (x,y,z system) and B(x', y', z') differ on positions/velocities, they still see $\vec{F} = m\vec{a}$.

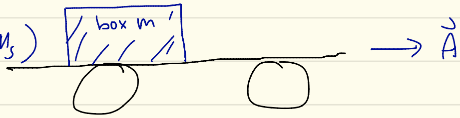
Case 2: $\ddot{\vec{R}} \neq 0$ (non-inertial frame) $\quad \vec{a} = \vec{A} + \vec{a}\,' \quad (\vec{A} \equiv \ddot{\vec{R}})$

$$\therefore\ m\vec{a}\,' = m\vec{a} - m\vec{A} \quad = \vec{F} + \vec{F}_{eff}$$

"Effective Force" $\vec{F}_{eff} = -m\vec{A}$ due to non-inertial frame ($\propto$ mass like gravity)

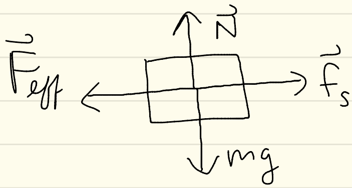
Example: accelerating train car

(coeff. of static friction μ_s)



Find largest $\vec{A}$ w/out the box sliding off

* Train rest frame non-inertial. We have



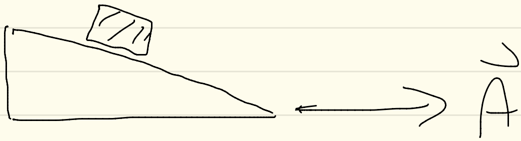
Friction: $f_s^{max} = \mu_s N = \mu_s mg$

* Box starts to slide if $F_{eff} > f_s^{max}$

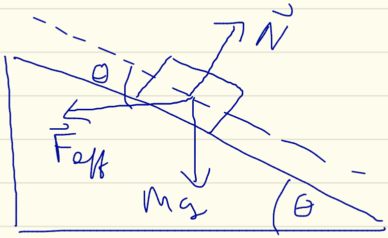
$\Rightarrow \cancel{m}A > \mu_s \cancel{m} g$

$\therefore A > \mu_s g$ the box will slide

Example 2: Mass on a pulled incline



* What $\vec{A}$ needed to make box slide up (No friction)?



* Block doesn't move if $F_{eff}\cos\theta = mg\sin\theta$

$\therefore$ If $F_{eff}\cos\theta > mg\sin\theta$, it will slide up.

$F_{eff} > mg\tan\theta$

or

$\cancel{m}A > \cancel{m}g\tan\theta$

$$\boxed{A > g\tan\theta}$$

Conservation Theorems in Newtonian Mechanics

* single particle/object: $\vec{F} = m\vec{a} = \frac{d}{dt}\vec{p} \quad (\vec{p} = m\vec{v})$

$$\boxed{\therefore \dot{\vec{p}} = 0 \text{ if no net force} \Rightarrow \vec{p} = \text{constant}}$$

* two particles exerting forces on eachother: (but no external forces)

$$\vec{F}_1 = -\vec{F}_2 \quad (3^{rd} \text{ Law})$$

$$\therefore \frac{d}{dt}\vec{p}_1 = -\frac{d}{dt}\vec{p}_2$$

$$\Rightarrow \frac{d}{dt}(\vec{p}_1 + \vec{p}_2) \equiv \frac{d}{dt}(\vec{p}_{TOT}) = 0$$

$$\boxed{\therefore \dot{\vec{P}}_{TOT} = 0 \text{ if no } \underline{\text{external}} \text{ forces acting on } 1+2.^{*} \Rightarrow \vec{P}_{TOT} = \text{constant}}$$

* In general, suppose $\vec{F} \cdot \hat{s} = 0$ for fixed unit vector $\hat{s}$ for a single particle

$$\Rightarrow \vec{F} \cdot \hat{s} = \frac{d\vec{p}}{dt} \cdot \hat{s} = 0 = \frac{d}{dt}(\vec{p} \cdot \hat{s})$$

$$\therefore \vec{p} \cdot \hat{s} = \text{constant} \quad //$$

* Likewise, consider 2 particles where $\vec{F}_1 \cdot \hat{s} = -F_2 \cdot \hat{s}$

$$\Rightarrow \frac{d}{dt}(\vec{P}_{TOT} \cdot \hat{s}) = 0 \Rightarrow \vec{P}_{TOT} \cdot \hat{s} = \text{constant} \quad //$$

* This assumes the internal forces obey Newton's 3^{rd} Law. Recall, there are "velocity-dependent" forces (ex: magnetic force between 2 moving point charges) where this is not the case. See the text for more discussion.

Cons. of Angular Momentum

$$\left.\begin{aligned} &\vec{L} \equiv \vec{r} \times \vec{p} \\ &\text{Torque } \vec{N} \equiv \vec{r} \times \vec{F} = \vec{r} \times \dot{\vec{p}} \end{aligned}\right\} \text{ defined wrt the origin}$$

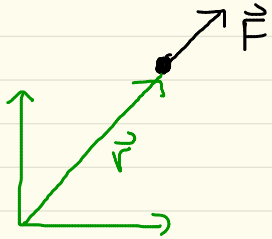
$$\therefore \frac{d\vec{L}}{dt} = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt} = \cancel{\vec{v} \times \vec{p}} + \vec{N}$$

0 since $\vec{p} = m\vec{v}$ and $\vec{v} \times \vec{v} = 0$

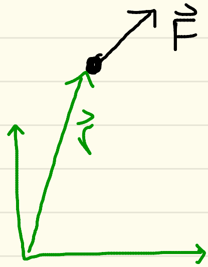
$$\boxed{\Rightarrow \dot{\vec{L}} = \vec{N}}$$

$\vec{L}$ = const. if particle not subjected to a torque

Dep. on coordinate choice



VS



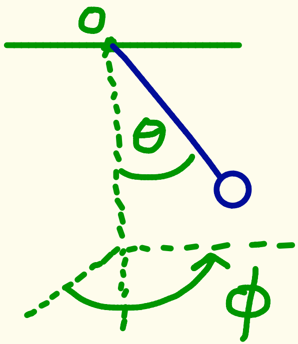
$\vec{N} = \vec{r} \times \vec{F} = 0$

$\dot{\vec{L}} = 0$

$\vec{N} = \vec{r} \times \vec{F} \neq 0$

$\dot{\vec{L}} \neq 0$

Clicker Quiz:



Consider a pendulum mass that moves such that both θ <u>and</u> ϕ are changing with time. The only forces at play are gravity + string tension.

During the motion, what can you say about $\vec{L}$ with respect to the suspension point O.

A.) total $\vec{L}$ is conserved

B.) No component of $\vec{L}$ is conserved

C.) Vertical component of $\vec{L}$ is conserved

D.) Horizontal component of $\vec{L}$ is conserved

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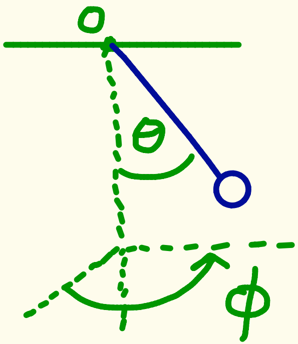

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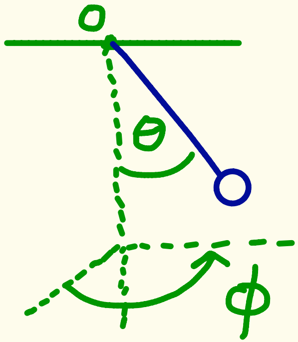

Solution

$$\frac{d\vec{L}}{dt} = \vec{N} = \vec{r} \times \vec{F}$$

$$\vec{F} = \vec{T} + \vec{F}_g$$

$$\vec{N}_T = \vec{r} \times \vec{T} = 0 \quad (\text{since } \vec{r} \parallel \vec{T})$$

$$\vec{N}_g = \vec{r} \times \vec{F}_g$$

by RH rule clearly $\vec{N}_g$ only has horizontal components.

$\Rightarrow$ No vertical $\vec{N}$-components $\Rightarrow$ L_{vert} conserved.

Work + Energy

$$\boxed{W_{12} \equiv \int_1^2 \vec{F}\cdot d\vec{r} \quad \text{work done by force } \vec{F} \text{ moving object from } 1 \to 2.}$$

$$\vec{F}\cdot d\vec{r} = m\frac{d\vec{v}}{dt}\cdot\frac{d\vec{r}}{dt}\,dt = m\frac{d\vec{v}}{dt}\cdot\vec{v}\,dt$$

*NOTE: Here $\vec{F}$ is the total (Net) force on particle.

$$= \frac{m}{2}\frac{d}{dt}(\vec{v}\cdot\vec{v})\,dt$$

$$= \frac{d}{dt}\left(\frac{m}{2}v^2\right)dt$$

$$= d\left(\frac{m}{2}v^2\right)$$

$$\Rightarrow \boxed{W_{12} = \int_1^2 \vec{F}\cdot d\vec{r} = \int_1^2 d\left(\frac{m}{2}v^2\right) = T_2 - T_1 \qquad \text{where } T \equiv \frac{1}{2}mv^2 = KE}$$

Conservative Forces + Potential Energy

* If $\vec{F} = -\vec{\nabla}U$ (*) (U = PE function. Here, we consider $U(\vec{r})$ or $U(\vec{r},t)$, but <u>NOT</u> $U(\vec{v})$.)

$$\Rightarrow W_{12} = -\int_1^2 \vec{\nabla}U\cdot d\vec{r} = -\int_1^2 dU = U_1 - U_2$$

(*) Necessary + Sufficient condition for $\vec{F} = -\vec{\nabla}U$ is $\vec{\nabla}\times\vec{F} = 0$

Total Energy: $E = T + U$

$$\frac{dE}{dt} = \frac{dT}{dt} + \frac{dU}{dt}$$

* but $dT = d\left(\frac{1}{2}mv^2\right) = \vec{F}\cdot d\vec{r}$ (see earlier)

$$\therefore \frac{dT}{dt} = \vec{F}\cdot\frac{d\vec{r}}{dt} = \vec{F}\cdot\dot{\vec{r}} \quad //$$

also have: $$\frac{dU}{dt} = \frac{\partial U}{\partial t} + \sum_{i=1}^{3} \frac{\partial U}{\partial x_i}\frac{dx_i}{dt}$$

$$= \frac{\partial U}{\partial t} + \vec{\nabla}U\cdot\dot{\vec{r}}$$

$$\therefore \frac{dE}{dt} = \underbrace{(\vec{F} + \vec{\nabla}U)}_{0}\cdot\dot{\vec{r}} + \frac{\partial U}{\partial t}$$

* Thus, for any system where $\vec{F} = -\vec{\nabla}U$ and $U = U(\vec{r}, t)$,

$$\frac{dE}{dt} = \frac{\partial U}{\partial t}$$

$\Rightarrow$ If $U = U(\vec{r})$, then $\dot{E} = 0$ & $E = T + U$ is constant

"Conservative Force"