Final Exam Review

* Wed. 5/1 3:00-5:00 pm (Nonprogrammable calculator)

* Exam is comprehensive

* I'll provide some formulas (see example exam)

* Here I list the review topics of material from exam 3 on
  (for topics on exams 1-3, see earlier review notes)
  posted on web page

1.) Leftover Central Force material not on exam 3

* material from §8.7 (Planetary Motion/ Kepler's Laws)

$$\theta(r) = \int \frac{\ell/r^2 \, dr}{\sqrt{2\mu\left(E + \frac{k}{r} - \frac{\ell^2}{2\mu r^2}\right)}} + \text{const}$$

$\Downarrow$ integrated

$$\boxed{\frac{\alpha}{r} = 1 + \varepsilon \cos\theta}$$

* normalized so $r(\theta=0) = r_{min}$

$$\alpha \equiv \frac{\ell^2}{\mu k}$$

$$\varepsilon = \sqrt{1 + \frac{2E\ell^2}{\mu k^2}}$$

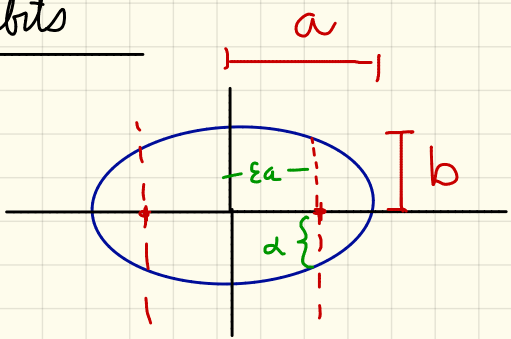
Types of conic sections for orbits

$$\left.\begin{array}{l} \text{①}\ \varepsilon>1 \quad (E>0) \Rightarrow \text{Hyperbola} \\ \text{②}\ \varepsilon=1 \quad (E=0) \Rightarrow \text{Parabola} \end{array}\right\} \text{unbound}$$

$$\left.\begin{array}{l} \text{③}\ 0<\varepsilon<1 \quad (V^{eff}_{min}<E<0) \Rightarrow \text{Ellipse} \\ \text{④}\ \varepsilon=0 \quad (E=V^{eff}_{min}) \Rightarrow \text{Circle} \end{array}\right\} \text{bound}$$

* Understand figs. 8-8 & 8-9 as to how these look

Elliptical Orbits



* Sun sits at focus

$$a = \frac{\alpha}{1-\varepsilon^2} = \frac{K}{2|E|}$$

$$b = \frac{\alpha}{\sqrt{1-\varepsilon^2}} = \frac{\ell}{\sqrt{2\mu|E|}}$$

$$r_{min} = a(1-\varepsilon) = \frac{\alpha}{1+\varepsilon}$$

$$r_{max} = a(1+\varepsilon) = \frac{\alpha}{1-\varepsilon}$$

Period $\tau = \pi K \sqrt{\frac{\mu}{2}}\, |E|^{-3/2} = \sqrt{\frac{4\pi^2 \mu}{K} a^3}$ (derive from $\frac{dA}{dt} = \frac{\ell}{2\mu}$)

Kepler's Laws

1. Planets move in ellipses w/ Sun at one of the foci

2. $\dfrac{dA}{dt} = \text{const} = \dfrac{\ell}{2\mu}$

3. Period $\tau^2 \propto a^3$

② Lagrangian Mechanics

* generalized coordinates (e.g., N-bodies => $s = 3N$ degrees of freedom)

$$L = T - U$$

Can write $\quad L = L(r_{\alpha,i}, \dot{r}_{\alpha,i}) \qquad \begin{matrix} \alpha = 1 \ldots N \\ i = x, y, z \end{matrix}$

or more generally,

$$L = L(q_j, \dot{q}_j) \quad j = 1 \ldots s \qquad (s = 3N \text{ here})$$

where $\quad q_j = q_j(r_{\alpha,i})$ and $r_{\alpha,i} = r_{\alpha,i}(q_1 \ldots q_s)$

"generalized coords"

Key Point: Lagrange eqns. of motion look the same whether using r_x, r_y, r_z or q_1, q_2, q_3

eg: 1 particle in 3d

Could use $L = L(x, y, z, \dot{x}, \dot{y}, \dot{z})$

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0 \quad (+ \text{ ditto for } y, z)$$

or

Could use $L = L(q_1, q_2, q_3, \dot{q}_1, \dot{q}_2, \dot{q}_3)$

$$\Rightarrow \frac{\partial L}{\partial q_1} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_1} = 0 \quad \text{etc}$$

* Hamilton's Principle

The physical trajectories $q_1(t), \cdots, q_s(t)$ of a system minimize the integral

$$J[q_1, q_2, \dots q_s] = \int_{t_1}^{t_2} L(q_1, \dots q_s; \dot{q}_1, \dot{q}_2, \dots \dot{q}_s; t)\, dt$$

Euler eqns. follow from

$$\frac{dJ}{d\alpha} = 0 \qquad \text{where} \qquad q_i(t;\alpha) = q_i(t) + \alpha\, \eta_i(t)$$

$$\eta_i(t_1) = \eta_i(t_2) = 0$$

* See book & class examples for setting up L in generalized coords

Cyclic Coordinates

If $\frac{\partial L}{\partial q_j} = 0 \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = 0$

"q_j is a cyclic coordinate"

$\Rightarrow \frac{\partial L}{\partial \dot{q}_j}$ = conserved (constant)

Generalized Force ; $Q_j \equiv \frac{\partial L}{\partial q_j}$

Generalized Momentum : $\tilde{p}_j = \frac{\partial L}{\partial \dot{q}_j}$

Probs. w/constraints

e.g., particle on some surface

$$L = L(q_i, \dot{q}_i) \quad i = 1 \ldots s$$

AND

$$f_K(q_i; t) = 0 \quad K = 1 \ldots m$$

* 2 ways to deal w/these problems

① Use the m constraint eqns to eliminate m of the q_i

$$L(q_1, q_2, \ldots q_s) \longrightarrow L(q_1, q_2, \ldots q_{s-m})$$

Solve: $$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} \quad i = 1 \ldots (s-m)$$

Easy, but no way to find force of constraint

② Lagrange Multipliers

* treat all s DOF as independent but solve modified Euler-Lagrange eqns & $f_k = 0$

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} + \sum_{k=1}^{m} \lambda_k(t)\frac{\partial f_k}{\partial q_i} = 0 \qquad i = 1 \ldots s$$

AND

$$f_k(q_1 \ldots q_s) = 0 \qquad k = 1 \ldots m$$

$(s+m)$ eqns for $s+m$ unknowns $(q_1 \ldots q_s)$ & $(\lambda_1, \ldots \lambda_m)$

$\lambda_k(t)$ = "Lagrange Multipliers"

Generalized Forces of Constraint

$$Q_j^{constraint} = \sum_{k=1}^{m} \lambda_k \frac{\partial f_k}{\partial q_j}$$