* Generalized Coordinates

N-bodies in 3d $\Rightarrow$ 3N degrees of freedom

e.g. $\{r_{\alpha,i}\}$ $\alpha = 1 \ldots N$, $i = x, y, z$ (cartesian coords)

$\{r_{\alpha,i}\}$ $\alpha = 1 \ldots N$, $i = r, \theta, \phi$ (spherical)

Key Point: Lagrange EOM take the same generic form

$$\frac{\partial L}{\partial r_{\alpha,i}} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}_{\alpha,i}}\right) = 0$$

* It's more general than this. ANY 3N-independent parameters ("generalized coords.") $\{q_1, q_2, \ldots q_{3N}\}$ (i.e., not just coordinate systems) can be used

* More Precise def. of Generalized Coords.

* for system w/ $s = 3N$ D.O.F.

$$r_{\alpha,i} = r_{\alpha,i}(q_1, q_2, \dots q_s, t)$$

$\alpha = 1 \dots N$
$i = 1,2,3$

$$\dot{r}_{\alpha,i} = \dot{r}_{\alpha,i}(q_1 \dots q_s, \dot{q}_1 \dots \dot{q}_s, t)$$

$\dot{q}_i$ = "generalized velocity"

* invertible functions

$$\Rightarrow \quad q_1 = q_1(r_{\alpha,i}, t) \quad \text{etc...}$$

$$\dot{q}_1 = \dot{q}_1(r_{\alpha,i}, \dot{r}_{\alpha,i}, t) \quad \text{etc.}$$

$$L(r_{\alpha,i}, \dot{r}_{\alpha,i}) = L(q_j, \dot{q}_j) \quad \text{(Scalar function)}$$

$$\Rightarrow \boxed{\frac{\partial L}{\partial q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} = 0 \qquad j = 1 \dots s}$$

* For cartesian coords, recall

$$\frac{\partial L}{\partial r_{\alpha,i}} = F_{\alpha,i} \qquad \begin{array}{l}\alpha = 1 \ldots N \\ i = x, y, z\end{array}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}_{\alpha,i}} = \frac{d}{dt} p_{\alpha,i}$$

* For generalized coords

$\frac{\partial L}{\partial q_j}$ = j^{th} component of "generalized force" $\quad j = 1 \ldots . S$

$\frac{\partial L}{\partial \dot{q}_j}$ = " " of generalized momentum "conjugate to q_j"

e.g: $L = \left(\frac{m}{2}\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2\right) - U(r)$ $\qquad \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$

=> Generalized momentum for θ is $\measuredangle$ momentum

Lagrangian Mechanics w/ Constraints

* How to derive eqns. of motion for system with constraints

* look @ simplest case with 2 d.o.f.

Hamilton's Principle: Minimize $J[q_1, q_2] = \int_{t_1}^{t_2} L(q_i; \dot{q}_i; t)\, dt$

but subject to constraint

$$f(q_1, q_2; t) = 0$$

* How do we need to modify our earlier recipe to minimize $J[q_1, q_2]$ when there wasn't an added constraint between q_1 + q_2?

* The root of the problem

as before, let $q_i(t;\alpha) = q_i(t) + \alpha\,\eta_i(t)$ where $\eta_i(t_1) = \eta_i(t_2) = 0$

$$\frac{\partial J}{\partial \alpha} = \int_{t_1}^{t_2} dt \left[\left(\frac{\partial L}{\partial q_1} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_1} \right)\eta_1(t) + \left(\frac{\partial L}{\partial q_2} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_2} \right)\eta_2(t) \right]$$

Unconstrained case: set $\frac{\partial J}{\partial \alpha} = 0$ & use $\eta_1(t)$ & $\eta_2(t)$ independent

$\Rightarrow \frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0 \quad i = 1, 2$ follows

* But here we have $f(q_1, q_2; t) = 0 \Rightarrow df = 0 = \left(\frac{\partial f}{\partial q_1}\eta_1(t) + \frac{\partial f}{\partial q_2}\eta_2(t) \right) d\alpha$

$$\Rightarrow \eta_2(t) = -\frac{\frac{\partial f}{\partial q_1}}{\frac{\partial f}{\partial q_2}}\,\eta_1(t)$$

NOT INDEPENDENT !!

Way # 1 to deal with Constraints

* Use constraint eqn $f(q_1, q_2; t) = 0$ to eliminate q_2 in terms of q_1 (or visa-versa)

$$L(q_1, \dot{q}_1, q_2, \dot{q}_2) \longrightarrow L(q_1, \dot{q}_1, q_2(q_1), \dot{q}_2(q_1, \dot{q}_1))$$

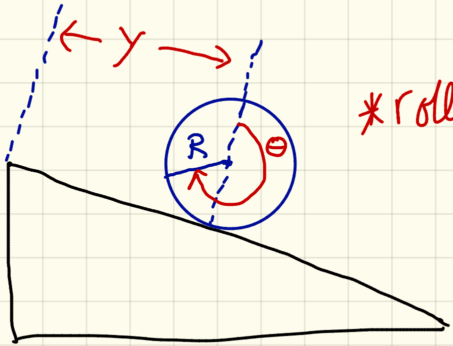
* Then Euler-Lagrange proceeds in terms of q_1 only

$$\frac{\partial L}{\partial q_1} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_1} = 0 \qquad \text{as before}$$

Pros: Simple. Don't ever refer to forces of constraint

Cons: If you want to determine $F_{\text{constraint}}$, you're out of luck.

Example:



*rolling disk without slipping

Generalized coordinates: $y + \theta$

Eqn. of Constraint: $y = R\theta$

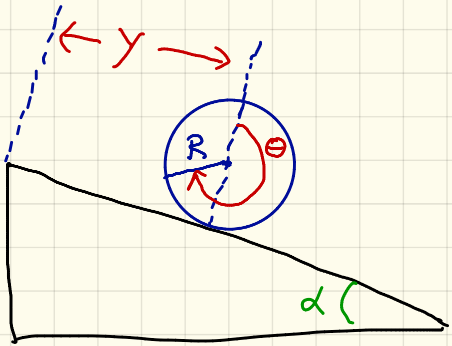
i.e., $y + \theta$ not independent due to the constraint of no slipping

$$T = T_{translation} + T_{rot}$$

$$= \frac{M\dot{y}^2}{2} + \frac{I\dot{\theta}^2}{2}$$

I = moment of inertia

$= \frac{1}{2}MR^2$ for disk



$$U = Mg(\ell - y)\sin\alpha$$

($*$ ℓ = length of incline)

$$\Downarrow$$

$$L = T - U = \frac{1}{2}M\dot{y}^2 + \frac{1}{4}MR^2\dot{\theta}^2 + Mg(y-\ell)\sin\alpha$$

$*$ Eqn of Constraint

$$f(y,\theta) = 0 = y - R\theta$$

$\Rightarrow$ eliminate $\theta = \dfrac{y}{R}$

$$\therefore L = \frac{1}{2}M\dot{y}^2 + \frac{1}{4}M\dot{y}^2 + Mg(y-\ell)\sin\alpha = \frac{3}{4}M\dot{y}^2 + Mg(y-\ell)\sin\alpha$$

OK, now that we've used $f(y,\theta)=0$ to eliminate θ-dep

$$\left.\begin{aligned} \frac{\partial L}{\partial y} &= Mg\sin\alpha \\ \frac{\partial L}{\partial \dot{y}} &= \frac{3}{2}M\dot{y} \end{aligned}\right\} \Rightarrow \quad \frac{\partial L}{\partial y} - \frac{d}{dt}\frac{\partial L}{\partial \dot{y}} = 0$$

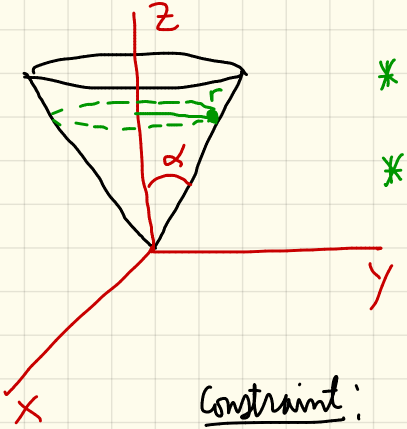
$$\Rightarrow \frac{3}{2}M\ddot{y} = Mg\sin\alpha$$

$$\Rightarrow \ddot{y} = \frac{2}{3}g\sin\alpha$$

$$\therefore \dot{y}(t) = \frac{2}{3}g\sin\alpha \cdot t$$

$$y(t) = \cancelto{0}{y(0)} + \frac{2}{3}g\sin\alpha \cdot \frac{t^2}{2} = \frac{g}{3}\sin\alpha\, t^2$$

* Another Constraint example using Way #1



* Particle constrained to move on inside of cone

* Derive EOM to find $r(t)$

Constraint: $\tan\alpha = \frac{r}{z}$

$\Rightarrow$ eliminate z for r (or visa versa) $\Rightarrow z = r\cot\alpha$

$\Rightarrow v^2 = \dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2 = \dot{r}^2 + r^2\dot{\theta}^2 + \dot{r}^2\cot^2\alpha$

$= \dot{r}^2(1+\cot^2\alpha) + r^2\dot{\theta}^2$

$v^2 = \dot{r}^2\csc^2\alpha + r^2\dot{\theta}^2$

$$\boxed{\Rightarrow T = \frac{m}{2}\left(\dot{r}^2\csc^2\alpha + r^2\dot{\theta}^2\right)}$$

* PE: $U = mgz = mgr\cot\alpha$

$$\Rightarrow L = \frac{m}{2}\left(\dot{r}^2\csc^2\alpha + r^2\dot{\theta}^2\right) - mgr\cot\alpha$$

* <u>EL eqn in θ</u>: $\frac{\partial L}{\partial\theta} = 0 \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) = 0 \Rightarrow mr^2\dot{\theta} = \text{constant}$

<u>General rule</u>: if $\frac{\partial L}{\partial q_i} = 0$, then generalized momentum $p_i = \frac{\partial L}{\partial\dot{q}_i}$ = conserved

* <u>EL eqn in r</u>:

$$\frac{\partial L}{\partial r} = mr\dot{\theta}^2 - mg\cot\alpha$$

$$\frac{\partial L}{\partial\dot{r}} = m\dot{r}\csc^2\alpha$$

$$\Rightarrow \boxed{\ddot{r}\csc^2\alpha - r\dot{\theta}^2 + g\cot\alpha = 0}$$

Way #2 to deal w/ Constraints: Lagrange Multipliers

$$\frac{\partial J}{\partial \alpha} = \int_{t_1}^{t_2} dt \left[\left(\frac{\partial L}{\partial q_1} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_1} \right) \eta_1(t) + \left(\frac{\partial L}{\partial q_2} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_2} \right) \eta_2(t) \right]$$