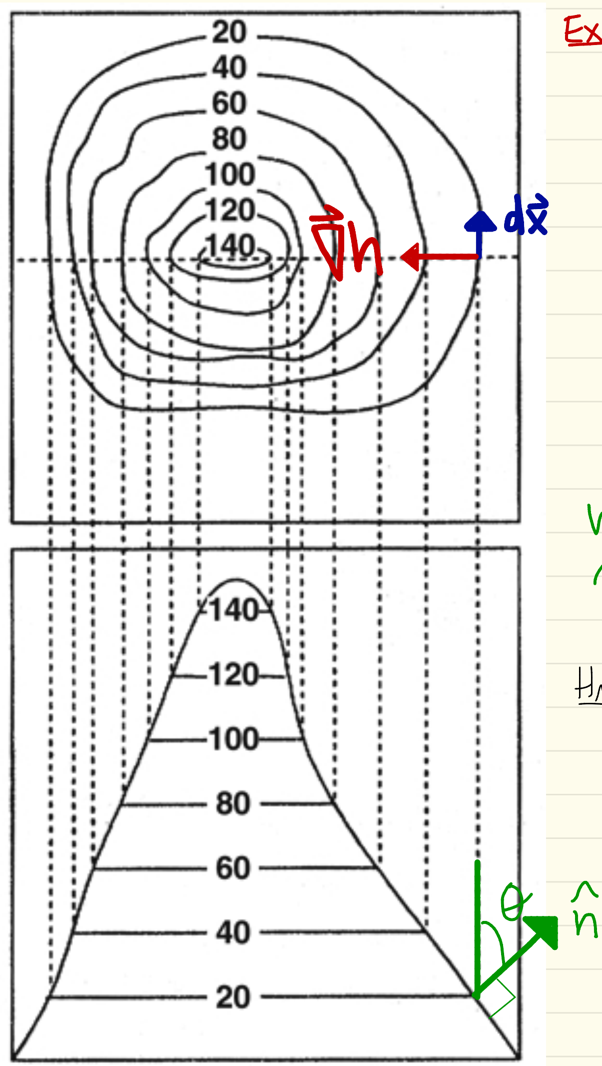
Gradient Recap from last time

$$\vec{\nabla} f(\vec{x}) = \sum_i \hat{e}_i \frac{\partial f}{\partial x_i} \iff df(\vec{x}) = \vec{\nabla} f(\vec{x}) \cdot d\vec{x} = |\vec{\nabla} f||d\vec{x}| \cos\theta$$



Ex: $z = h(x,y)$

$\vec{\nabla} h(x,y) \perp$ curve of constant $h(x,y)$.

(direction of steepest change)

HW1 Question
What angle does $\hat{n}$ make w/ the z-axis?

Hint: $\vec{\nabla} h(x,y) \perp$ to 2d curve $h(x,y) = \text{const}$.

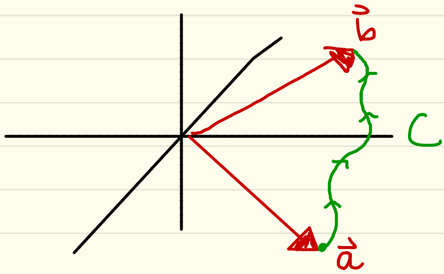
$\vec{\nabla} f(x,y,z) \perp$ to 3d surface of $f(x,y,z) = \text{constant}$

Analog of FTC for gradients

FTC for 1d: $\int_a^b \frac{df}{dx}\, dx = f(b) - f(a)$

3d is richer since there are 3 types of derivatives $\vec{\nabla} f$, $\vec{\nabla}\cdot\vec{F}$, $\vec{\nabla}\times\vec{F}$

FTC for gradients: $\int_C \vec{\nabla} f(\vec{x}) \cdot d\vec{x}$ "line integral"



* but $\vec{\nabla} f(\vec{x}) \cdot d\vec{x} = df(\vec{x})$

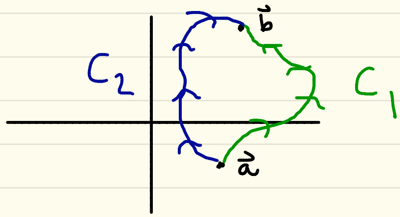
$$\Rightarrow \boxed{\int_C \vec{\nabla} f \cdot d\vec{x} = \int_C df(\vec{x}) = f(\vec{b}) - f(\vec{a})}$$

Corollaries: 1) $\int_{C(\vec{a},\vec{b})} \vec{\nabla} f \cdot d\vec{x} = \int_{C'(\vec{a},\vec{b})} \vec{\nabla} f \cdot d\vec{x}$

"Path-independent"

② $\oint_C \vec{\nabla} f(\vec{x}) \cdot d\vec{x} = 0$

proof: let the closed contour C be $C = C_1 - C_2$



$$\oint_C \vec{\nabla} f \cdot d\vec{x} = \int_{C_1} \vec{\nabla} f \cdot dx - \int_{C_2} \vec{\nabla} f \cdot d\vec{x}$$

$$= 0 \text{ since } \int_{C_1} = \int_{C_2}.$$

Corollaries 1+2 important for "conservative forces"

$$\vec{F} = -\vec{\nabla} U$$

PE function

The Divergence of a Vector Field

* Since $\vec{\nabla} f = \sum_{i=1}^{3} \hat{e}_i \frac{\partial f}{\partial x_i}$, it's convenient to define the "del" or "nabla" operator

$$\boxed{\vec{\nabla} \equiv \hat{e}_1 \frac{\partial}{\partial x_1} + \hat{e}_2 \frac{\partial}{\partial x_2} + \hat{e}_3 \frac{\partial}{\partial x_3}}$$

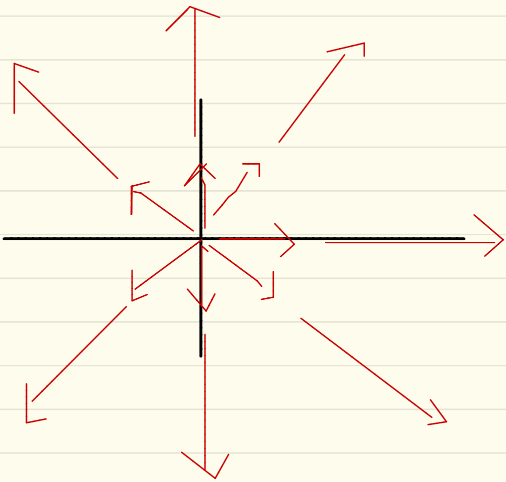
* treat it like a vector w/ the caveat that you have to "feed" it something to act on

$$\boxed{\text{Divergence: } \vec{\nabla} \cdot \vec{F}(\vec{x}) = \sum_{i=1}^{3} \frac{\partial F_i}{\partial x_i}}$$

(just like $\vec{A} \cdot \vec{B} = \sum_{i=1}^{3} A_i B_i$)

Cartesian only!

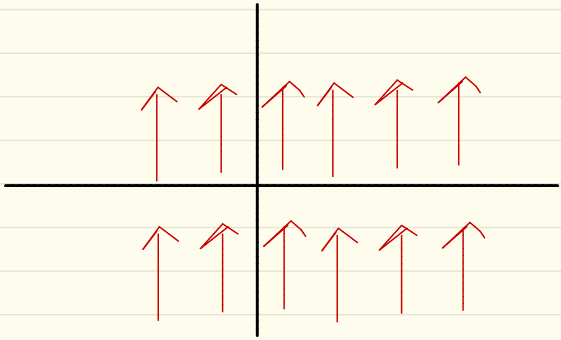
Example 1: $\vec{F} = \vec{x}$



$$\vec{\nabla} \cdot \vec{F} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3$$

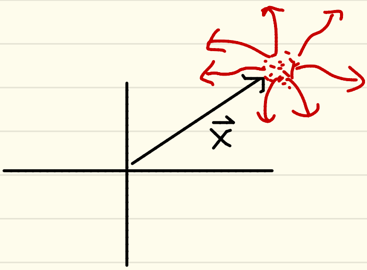
Example 2: $\vec{F}(\vec{x}) = C\hat{j} \qquad C = \text{const}$

$\vec{\nabla}\cdot\vec{F} = 0$ by inspection



Physical meaning of $\vec{\nabla}\cdot\vec{F}$

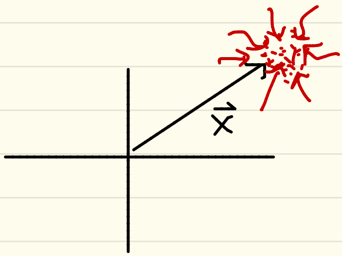
* Imagine $\vec{F}(\vec{x})$ is the velocity profile of H_2O in a river
* Sprinkle sawdust at $\vec{x}$



$\vec{\nabla}\cdot\vec{F}(\vec{x}) > 0$

Spreads out from $\vec{x}$

"Source at $\vec{x}$"



$\vec{\nabla}\cdot\vec{F}(\vec{x}) < 0$

Converges towards $\vec{x}$

"Sink at $\vec{x}$"

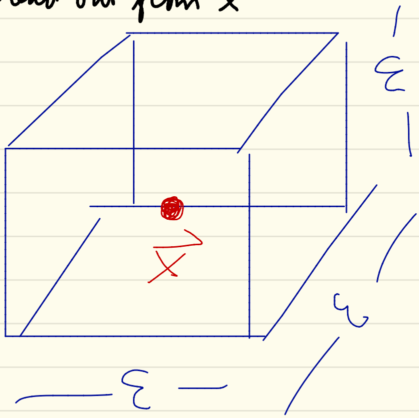
Physical Derivation of $\vec{\nabla}\cdot\vec{F}$

* Consider fluid w/ uniform ρ + velocity $\vec{V}(\vec{x})$ profile

Let $\vec{F}(\vec{x}) = \rho\vec{V}(\vec{x})$ $\qquad [\rho] = \frac{kg}{m^3}$, $[\vec{V}] = \frac{m}{s}$

$$[\vec{F}] = \frac{kg}{m^2 \cdot s} = \frac{kg}{\text{area} \cdot s}$$

How does the fluid spread out from $\vec{x}$?



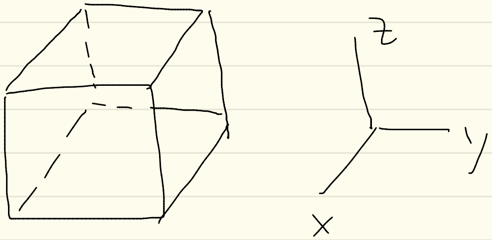
* one way to answer this is to measure the rate of mass influx/outflow thru the walls of the tiny cube.

(d$\vec{A}$ outward normal)

$$\text{Flux}\left(\frac{kg}{s}\right) \equiv \oint_{\partial S} \vec{F}\cdot d\vec{A} = \sum_{i=1}^{6} \int \vec{F}(\vec{x})\cdot d\vec{A}(i)$$

6 faces of the cube

* To see how it works, I just do front & back faces of the cube



$d\vec{A}\,(\text{front}) = dy\,dz\,\hat{\imath}$ and $d\vec{A}\,(\text{back}) = -dy\,dz\,\hat{\imath}$

$$\int \vec{F}(\vec{x})\cdot d\vec{A}_{\text{front}} = \int_{z-\varepsilon/2}^{z+\varepsilon/2} dz \int_{y-\frac{\varepsilon}{2}}^{y+\varepsilon/2} dy\; F_x\!\left(\vec{x}+\tfrac{\varepsilon}{2}\hat{\imath}\right) \approx \varepsilon^2 F_x\!\left(\vec{x}+\tfrac{\varepsilon}{2}\hat{\imath}\right) = \varepsilon^2 F_x\!\left(x+\tfrac{\varepsilon}{2},y,z\right)$$

$$\left(\text{I used } \int_{a-\varepsilon/2}^{a+\varepsilon/2} f(x)\,dx \approx \varepsilon f(a) \quad \text{as } \varepsilon\to 0\right)$$

likewise, $$\int_{\text{back}} \vec{F}\cdot d\vec{A} = -\int_{z-\varepsilon/2}^{z+\varepsilon/2} dz \int_{y-\varepsilon/2}^{y+\varepsilon/2} dy\; F_y\!\left(\vec{x}-\tfrac{\varepsilon}{2}\hat{\imath}\right) \approx -\varepsilon^2 F_x\!\left(\vec{x}+\tfrac{\varepsilon}{2}\hat{\imath}\right) = -\varepsilon^2 F_x\!\left(x-\tfrac{\varepsilon}{2},y,z\right)$$

$$\therefore \int_{\text{front}} + \int_{\text{back}} = \varepsilon^2\left(F_x\!\left(x+\tfrac{\varepsilon}{2},y,z\right) - F_x\!\left(x-\tfrac{\varepsilon}{2},y,z\right)\right)$$

$$\approx \varepsilon^2\left(\cancel{F_x(x,y,z)} + \frac{\varepsilon}{2}\frac{\partial F_x(x,y,z)}{\partial x} + O(\varepsilon^2) - \cancel{F_x(x,y,z)} + \frac{\varepsilon}{2}\frac{\partial F_x(x,y,z)}{\partial x} + O(\varepsilon^2)\right)$$

$$= \varepsilon^3 \frac{\partial F_x(x,y,z)}{\partial x}$$

Doing analogous calculations for other 4 sides + adding them up:

$$\Rightarrow \quad \text{Rate of mass flow in/out of cube} = \oint_{\partial S} \vec{F}\cdot d\vec{A} = \varepsilon^3\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\right)$$

$$= \varepsilon^3\, \vec{\nabla}\cdot\vec{F}(\vec{x})$$

Since $\varepsilon^3 = \delta V$ (volume of infinitesimal cube at $\vec{x}$), we have

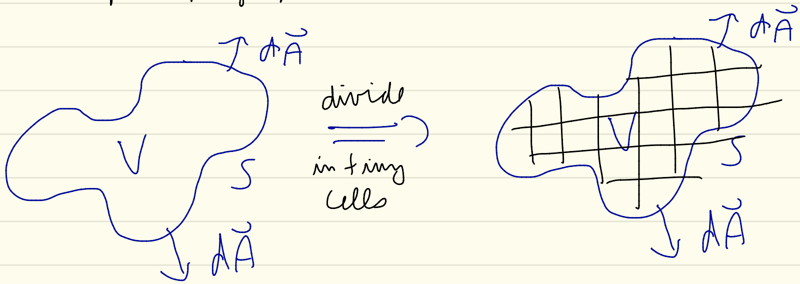
$$\boxed{\vec{\nabla}\cdot\vec{F}(\vec{x}) = \lim_{\delta V\to 0} \frac{1}{\delta V}\oint_{\partial S} \vec{F}\cdot d\vec{A}}$$

* FTC for $\vec{\nabla}\cdot\vec{F}$ – Gauss's Theorem (aka Divergence Theorem)

$$\boxed{\int_V \vec{\nabla}\cdot\vec{F}(\vec{x})\, d^3x = \oint_S \vec{F}\cdot d\vec{A}}$$

(volume) (closed bounding surface of V)

* Sketch of the proof of Gauss's Thm.

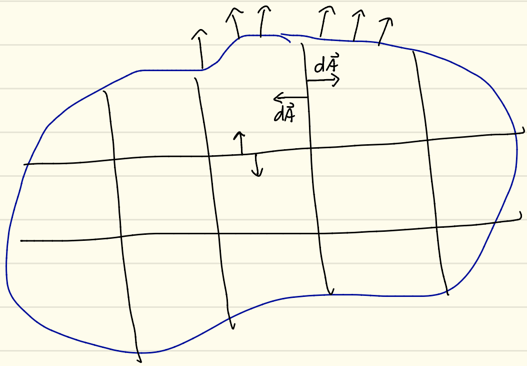


$$V = \sum_i \delta V_i$$

$$\int_V \vec{\nabla}\cdot\vec{F}\, d^3x = \sum_i \int_{V_i} \vec{\nabla}\cdot\vec{F}\, d^3x \approx \sum_i \delta V_i \left(\vec{\nabla}\cdot\vec{F}\right)$$

* but recall, $\vec{\nabla}\cdot\vec{F} \equiv \lim_{\delta V \to 0} \frac{1}{\delta V} \oint_{\delta S} \vec{F}\cdot d\vec{A}$

$$\Rightarrow \sum_i (\vec{\nabla}\cdot\vec{F})\,\delta V_i = \sum_i \oint_{\delta S_i} \vec{F}\cdot d\vec{A}_{(i)}$$



$\Rightarrow$ shared cell walls cancel leaving just surface integral over outer surface.

$$\sum_i \oint_{\delta S_i} \vec{F}\cdot d\vec{A} = \oint_S \vec{F}\cdot d\vec{A} \quad \underline{\text{QED}}$$