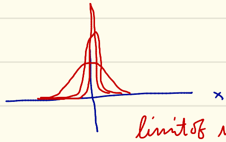
* Recap from last time

Dirac delta function:

$$\delta(x) = \begin{cases} 0 & x \neq 0 \\ \infty & x = 0 \end{cases}$$



limit of infinitely narrow spike @ $x=0$ but w/ unit area

where $\displaystyle\int_a^b \delta(x)\,dx = 1$ where $a < 0 < b$

$$\int_a^b \delta(x) f(x)\,dx = f(0)$$

arbitrary function

Shifted δ-fn:

$$\delta(x-x') = \begin{cases} 0 & x \neq x' \\ \infty & x = x' \end{cases} \Rightarrow \int_a^b \delta(x-x')\,dx = 1 \quad \text{if} \quad a < x' < b$$

$$\int_a^b \delta(x-x') f(x)\,dx = f(x')$$

3d δ-fn:

$$\delta^3(\vec{r}) \equiv \delta(x)\delta(y)\delta(z)$$

$$\int_{Vol} \delta^3(\vec{r}) F(\vec{r})\,d^3r = F(0) \quad \text{if Vol encloses } \vec{r}=0$$
$$= 0 \quad \text{else}$$

Useful identities we derived

$$\vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right) = 4\pi\,\delta^3(\vec{r})$$

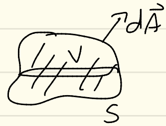
$$\vec{\nabla}\cdot\left(\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}\right) = 4\pi\,\delta^3(\vec{r}-\vec{r}')$$

Used these to derive Gauss's Law

Gauss's Law (Differential + Integral)

$$\boxed{\vec{\nabla}\cdot\vec{g}(\vec{r}) = -4\pi G\,\rho(\vec{r})}$$

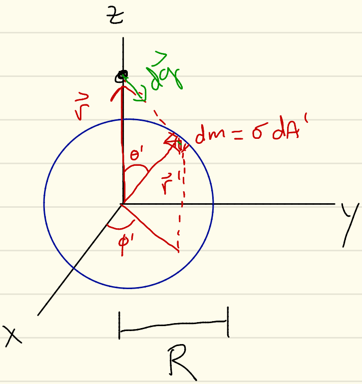
✪
$$\boxed{\begin{aligned} \oint_S \vec{g}\cdot d\vec{A} &= -4\pi G\, M_{enclosed} \\ M_{enclosed} &= \int_V d^3r\, \rho(\vec{r}) \end{aligned}}$$



Eqn ✪ is true in general. However, in specific cases where $\rho(\vec{r})$ is highly symmetric, then ✪ gives us a way to trivially calculate $\vec{g}(\vec{r})$.

To appreciate the labor saved by ✪, let's first sketch the sol'n for a problem in the "brute force" approach.

e.g.: Hollow spherical shell w/ uniform surface $\sigma = \dfrac{M}{4\pi R^2}$



$$d\vec{g}(\vec{r} = z\hat{k}) = -G\,dm\,\frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}$$

Symmetry: $\int d\vec{g}(0,0,z) = g(0,0,z)\,\hat{k}$

$$\therefore\; g = \int d\vec{g}\cdot\hat{k}$$

Identifying the relevant variable

$$\vec{r} = z\hat{k}$$

$$\vec{r}' = R\sin\theta'\cos\phi'\,\hat{\imath} + R\sin\theta'\sin\phi'\,\hat{\jmath} + R\cos\theta'\,\hat{k}$$

$$\Downarrow$$

$$|\vec{r}-\vec{r}'|^3 = \left[R^2 + z^2 - 2Rz\cos\theta'\right]^{3/2}$$

$$\therefore\ d\vec{g} = -G\,dm\,\frac{\left(R\sin\theta'\cos\phi'\,\hat{\imath} + R\sin\theta'\sin\phi'\,\hat{\jmath} + (z-R\cos\theta')\hat{k}\right)}{\left[R^2+z^2-2Rz\cos\theta'\right]^{3/2}}$$

$$\boxed{d\vec{g}\cdot\hat{k} = -G\,dm\,\frac{(z-R\cos\theta')}{\left[R^2+z^2-2Rz\cos\theta'\right]^{3/2}}}$$

* recall, $dA' = R^2\sin\theta'\,d\theta'\,d\phi'$ for spherical surface.

and

$$dm = \sigma\,dA' = \frac{M}{4\pi \cancel{R^2}}\cdot \cancel{R^2}\sin\theta'\,d\theta'\,d\phi'$$

$$\therefore\ g(0,0,z) = -\frac{MG}{4\pi}\int_0^{2\pi}\int_0^{\pi}\frac{(z-R\cos\theta')}{\left[R^2+z^2-2Rz\cos\theta'\right]^{3/2}}\sin\theta'\,d\theta'\,d\phi'$$

$$= -\frac{MG}{2}\int_0^{\pi}\frac{z-R\cos\theta'}{\left[R^2+z^2-2Rz\cos\theta'\right]^{3/2}}\sin\theta'\,d\theta'$$

* let $u=\cos\theta'$, $du = -\sin\theta'\,d\theta'$ $\Big\}\Rightarrow g = -\frac{MG}{2}\int_{-1}^{+1}\frac{(z-Ru)\,du}{\left[R^2+z^2-2Rzu\right]^{3/2}}$

$$g = -\frac{MG}{2}\int_{-1}^{+1}\frac{(z-R\mu)\,d\mu}{\left[R^2+z^2-2Rz\mu\right]^{3/2}}$$

$\Rightarrow$ (Thanks to Mathematica)

$$\text{integral} = \begin{cases} 0 & \text{if } z<R \\ \dfrac{2}{z^2} & \text{if } z\geq R \end{cases}$$

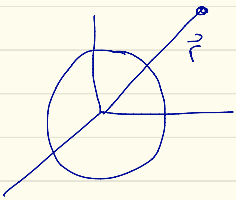
$\Rightarrow$ At the end of a long + tedious calculation, we thus find

$$\boxed{\vec{g}(0,0,z) = \begin{cases} 0 & \text{if } z<R \\ -\dfrac{MG}{z^2}\,\hat{k} & \text{if } z\geq R \end{cases}}$$

NOTE: for $z\geq R$, this is the same you would get for a point mass M placed at the origin.

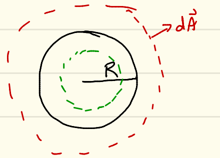
* Also note that the spherical symmetry implies

$$\vec{g}(\vec{r}) = \begin{cases} 0 & \text{if } |\vec{r}|<R \\ -\dfrac{MG}{r^2}\,\hat{r} & \text{if } |\vec{r}|\geq R \end{cases}$$



* OK, having convinced you that the Hollow sphere problem is somewhat tedious (& yet such a simple final form), we now show how to solve it trivially using Gauss's Law.

$$\oint_S \vec{g}\cdot d\vec{A} = -4\pi G M_{enc}$$



* the surface S is arbitrary. Let us consider S = sphere of radius r & consider the 2 cases $r > R$ & $r < R$.

Step 1: spherical symmetry (Mathematically, this means $\rho(r,\theta,\phi) = \rho(r)$. No directional dep!) implies

$$\vec{g}(r,\theta,\phi) = g(r)\,\hat{r}$$

(This is the crucial step in all Gauss's Law problems!)

Step 2: $d\vec{A} = r^2 \sin\theta\, d\theta\, d\phi\, \hat{r} \equiv r^2 d\Omega\, \hat{r}$

$$\therefore \oint_S \vec{g}\cdot d\vec{A} = \int_{\substack{\text{Sphere}\\ \text{radius } r}} g(r)\, r^2 d\Omega = g(r)\, 4\pi r^2$$

Case 1: $r < R \Rightarrow M_{enclosed} = 0$

$$\therefore g(r)\, 4\pi r^2 = 0 \Rightarrow g(r<R) = 0$$

Case 2: $r \geq R \Rightarrow M_{enclosed} = M \Rightarrow g(r>R) = \dfrac{-4\pi G M}{4\pi r^2} = -\dfrac{GM}{r^2}$

Which agrees w/ our earlier result!

General Idea: for very symmetric ρ, use symmetry to "guess" the form of $\vec{g}$
that let's you pull $\vec{g}$ out of the integral

$$\oint_S \vec{g}\cdot d\vec{A} \qquad \text{for suitably chosen } S. \text{ ("Gaussian Surface")}$$

* Can do this for 2 other "classes" of ρ
  - cylindrical symmetry
  - planar symmetry

Cylindrical (aka axial) Symmetry

Mathematical criteria $\rho(r,\phi,z) = \rho(r)$ NOTE: Here r,ϕ,z are cylindrical coords!

$$\boxed{\Rightarrow \vec{g}(r,\phi,z) = g(r)\,\hat{r}}$$

Example:

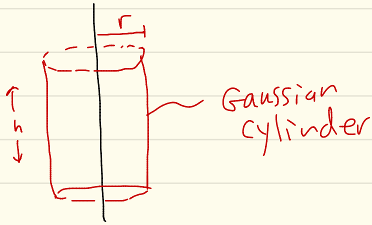
* brute force calc. for finite wire (length 2ℓ) done in class last week

- in midplane, argued that vertical components of $\vec{g}$ cancel. I.e.,

$$\vec{g}_{2\ell}(r) = -G\lambda r \cdot \frac{2\ell}{r^2\sqrt{\ell^2+r^2}}\,\hat{r} \qquad \text{in } z=0 \text{ plane}$$

* for ∞-wire, every point is the "midplane" (not just $z=0$)

$$\Rightarrow \vec{g}(r,\phi,z) = g(r)\,\hat{r}$$



$$\oint \vec{g}\cdot d\vec{A} = \underbrace{\cancel{\int \vec{g}\cdot d\vec{A}}}_{\text{Lids of cylinder}} + \underbrace{\int \vec{g}\cdot d\vec{A}}_{\text{sides of cylinder}}$$

0 since $d\vec{A} \perp \vec{g}$

for sides of cylinder, $d\vec{A} = \hat{r}\,2\pi r\,dz$

$$\Rightarrow \oint \vec{g}\cdot d\vec{A} = g(r)2\pi r \int_{-\frac{h}{2}}^{h/2} dz = g(r)\,2\pi r h$$

$$M_{enclosed} = h\lambda$$

$$\therefore \oint \vec{g}\cdot d\vec{A} = -4\pi G M_{enc} \Rightarrow g(r)2\pi r\cancel{h} = -4\pi G\cancel{h}\lambda$$

$$\Downarrow$$

$$\boxed{g(r) = \frac{-2G\lambda}{r}}$$

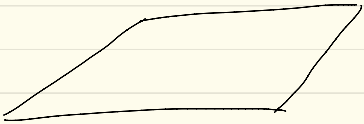
* Check it by comparing to $\ell \to \infty$ result derived for finite wire

$$\vec{g}_{2\ell}(r) = -G\lambda r \cdot \frac{2\ell}{r^2\sqrt{\ell^2+r^2}}\,\hat{r} \qquad \text{in } z=0 \text{ plane}$$

$$= \frac{-G\lambda 2\ell}{r\ell\sqrt{1+r^2/\ell^2}} \xrightarrow{\ell\to\infty} \frac{-2G\lambda}{r} \quad \checkmark.$$

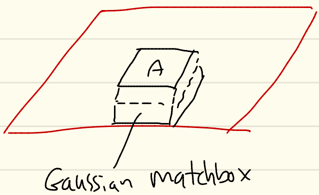
NOTE: For cylindrical symmetric masses, strictly speaking they must be infinite in the z-direction for $\rho(r,\phi,z) = \rho(r)$ to be obeyed. HOWEVER, we often cheat & assume $\vec{g} = g(r)\,\hat{r}$ if the mass is "very long" but finite. In this case, the Gauss's law result will be the most accurate close to the midplane $z=0$.

Planar Symmetry example



∞-plane w/ uniform σ at $z=0$

Symmetry: $\vec{g} = \begin{cases} -|g|\,\hat{k} & z>0 \\ +|g|\,\hat{k} & z<0 \end{cases}$



$$\oint_{\text{Matchbox}} \vec{g}\cdot d\vec{A} = \int_{\text{Top}} \vec{g}\cdot d\vec{A} + \int_{\text{bottom}} \vec{g}\cdot d\vec{A} + \cancel{\int_{\text{Sides}} \vec{g}\cdot d\vec{A}}^{\,0}$$

$$d\vec{A}_{top} = dxdy\,\hat{k}$$
$$d\vec{A}_{bot} = -dxdy\,\hat{k}$$

$$\therefore \oint \vec{g}\cdot d\vec{A} = -2|g|A = -4\pi G M_{enc}$$

$$M_{enc} = A\sigma$$

$$\Rightarrow -2|g|\cancel{A} = -4\pi G\cancel{A}\sigma$$

$$\Rightarrow \boxed{|g| = 2\pi\sigma G}$$