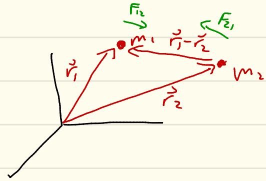
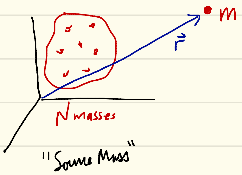
* Recap of Newton's Universal Law of Gravitation (ch 5)

$$\boxed{\vec{F}_{12} = -G m_1 m_2 \frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|^3}} \quad ①$$



$G = 6.673 \times 10^{-11} \ \frac{N \cdot m^2}{kg^2}$ "Universal Gravitation Constant"

* Superposition Principle



$$\boxed{\vec{F}(m \text{ at } \vec{r}) = -\sum_{i=1}^{N} G m m_i \frac{\vec{r} - \vec{r}_i}{|\vec{r} - \vec{r}_i|^3}} \quad ②$$

* Gravitational Field Concept

re-write ② as

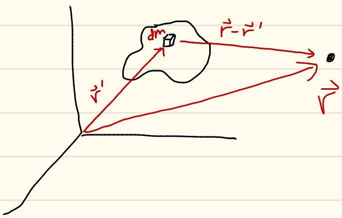
$$\vec{F}(m \text{ at } \vec{r}) = m \vec{g}(\vec{r})$$

$$\vec{g}(\vec{r}) = -\sum_{i=1}^{N} G m_i \frac{\vec{r} - \vec{r}_i}{|\vec{r} - \vec{r}_i|^3} \quad \text{"Gravitational field"} \quad ③$$

* View the N "source masses" as creating a condition of space (i.e., $g(\vec{r})$) such that if a mass is placed at $\vec{r}$, it feels a force by eq. ③

Note: Here (+ the analogous introduction of the Electric field concept to re-write Coulomb's Law), the gravitational field $\vec{g}(\vec{r})$ seems like a trivial re-writing of the more fundamental force equations (① + ②). However, the field concept in the end frees us from a troublesome "action at a distance" picture of forces. You'll learn in E+M and CM II how the field concept allows a harmonious description consistent with Einstein's Special Relativity.

* Generalization to Continuous mass distribution



$$d\vec{g}(\vec{r}) = -G\, dm \, \frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}$$

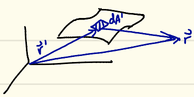
$* \; dm = \rho(\vec{r}\,')\, d^3r'$

$$\Downarrow$$

$$\boxed{g(\vec{r}) = -G \int d^3r' \, \rho(\vec{r}\,') \, \frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}}$$

3d mass distribution

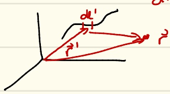
* obvious generalization to 1D + 2D



$dm = \sigma(\vec{r}\,')\, dA'$

$$g(\vec{r}) = -G \int dA' \, \sigma(\vec{r}\,') \, \frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}$$

$dm = \lambda(\vec{r}\,')\, dl'$



$$g(\vec{r}) = -G \int dl' \, \lambda(\vec{r}\,') \, \frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}$$

Tricks for finding $\vec{g}(\vec{r})$ given $\rho(\vec{r})$ [Gauss's Law]

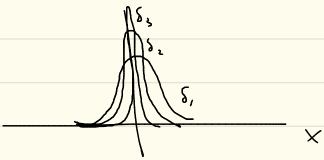
1st, let's do a crash course review of the Dirac Delta function which will prove useful to derive Gauss's Law. See any Math Methods book (e.g., Arfken or Boas) for details.

* 1d Dirac Delta function

$$\boxed{\delta(x) = \begin{cases} 0 & x \neq 0 \\ \infty & x = 0 \end{cases} \qquad \underline{\text{AND}} \qquad \int_{-\infty}^{\infty} \delta(x)\,dx = 1}$$

__Note__: $\delta(x)$ __NOT__ a well-behaved function in the usual sense. Mathematicians call it a distribution or a generalized function

* Can view $\delta(x)$ as the limit of a sequence of sharply peaked (but "normal") functions



$$\boxed{\delta(x) = \lim_{n\to\infty} \delta_n(x) \quad \text{where} \quad \int_{-\infty}^{\infty} \delta_n(x)\,dx = 1}$$

__ex1__: $\delta_n(x) = \frac{n}{\sqrt{\pi}} e^{-n^2x^2}$

__ex2__: $\delta_n(x) = \frac{\sin nx}{nx}$

__ex3__: $\delta_n(x) = \frac{n}{\pi} \frac{1}{1+n^2x^2}$

__Claim__: $\lim_{n\to\infty} \int_{-\infty}^{\infty} \delta_n(x) f(x)\,dx = f(0)$

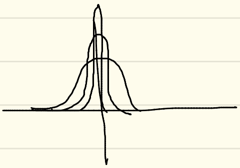
for arbitrary "well-behaved" $f(x)$

$$\boxed{\therefore \int_{-\infty}^{\infty} \delta(x) f(x)\,dx = f(0)}$$

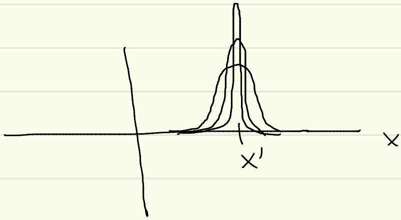
* Trivial Extensions

① $\int_{-\infty}^{\infty} \delta(x) f(x)\,dx = \int_{-a}^{b} \delta(x) f(x)\,dx = f(0)$ if $b>0$, $-a<0$

② "Shifted" δ-function



$\delta_n(x)$



$\delta_n(x-x')$

$$\Rightarrow \int_{-\infty}^{\infty} dx\, \delta(x-x') f(x) = f(x')$$

$$= \int_a^b dx\, \delta(x-x') f(x) \quad (\text{if } a \le x' \le b)$$

③ 3d - delta function $\delta^{(3)}(\vec{r}) \equiv \delta(x)\delta(y)\delta(z)$

$\Downarrow$

$$\int_{\substack{\text{Vol enclosing}\\ \text{origin}}} \delta^{(3)}(\vec{r})\, d^3r = 1$$

$$\int_{\substack{\text{Vol enc.}\\ \text{origin}}} \delta^{(3)}(\vec{r})\, F(\vec{r})\, d^3r = F(0,0,0)$$

Example: For N point masses $m_1, \dots m_N$, show $\rho(\vec{r}') = \sum_{i=1}^{N} m_i \, \delta^{(3)}(\vec{r}' - \vec{r}_i)$ gives the correct expression for $g(\vec{r})$

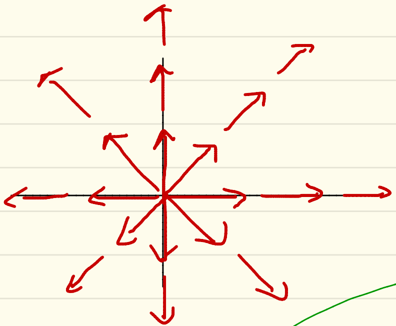
proof:
$$g(\vec{r}) = -G \int d^3r' \rho(\vec{r}') \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}$$

$$= -G \sum_{i=1}^{N} m_i \int d^3r' \, \delta^{(3)}(\vec{r}' - \vec{r}_i) \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}$$

$$= -G \sum_{i=1}^{N} m_i \frac{\vec{r}-\vec{r}_i}{|\vec{r}-\vec{r}_i|^3}$$ which agrees w/the discrete expression.

* Now let's ask an innocent question. What is $\vec{\nabla} \cdot \left(\frac{\hat{r}}{r^2}\right)$?

* sketch $\vec{F}(\vec{r}) = \frac{\hat{r}}{r^2}$ like we did in our math review during week 1.



Spreads out. So $\vec{\nabla} \cdot \vec{F} > 0$?

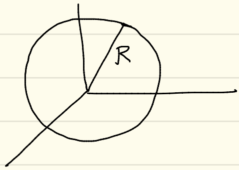
NO!

direct calculation

$$\vec{\nabla} \cdot \vec{F} = \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{1}{r^2}\right) = 0$$

(I used the spherical coord. expression for $\vec{\nabla} \cdot \vec{F}$. You of course get the same result (w/more algebra) using cartesian)

* It gets weirder still! Consider



$$\int_V \vec{\nabla}\cdot\vec{F}\, d^3r \overset{\text{Divergence}}{\underset{\text{thm.}}{=}} \oint_S \vec{F}\cdot d\vec{A}$$

$$\overset{(?)}{=} 0 \quad \text{since } \vec{\nabla}\cdot\vec{F} = 0$$

* Flux integral over the sphere: $\quad d\vec{A} = R^2 \sin\theta\, d\theta\, d\phi\, \hat{r}$

$$\vec{F}\cdot d\vec{A} = \frac{\hat{r}}{R^2}\cdot d\vec{A}$$

$$\therefore \oint \vec{F}\cdot d\vec{A} = \int_0^{\pi}\int_0^{2\pi} \frac{1}{\cancel{R^2}} \cancel{R^2} \sin\theta\, d\theta\, d\phi \quad = 4\pi \neq 0 \;!!$$ (For any sphere w/ $R>0$)

Problem: $\vec{F} = \frac{\hat{r}}{r^2}$ singular at $\vec{r}=0$, so $\vec{\nabla}\cdot\vec{F}$ is singular there too. In fact, all the apparent inconsistencies are resolved by noting

$$\boxed{\vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right) = 4\pi\, \delta^{(3)}(\vec{r})}$$

Trivial extension:

$$\boxed{\vec{\nabla}\cdot\left(\frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}\right) = 4\pi\, \delta^{(3)}(\vec{r}-\vec{r}\,')}$$

(hint: Use the chain rule $\frac{\partial}{\partial x} = \frac{\partial(x-x')}{\partial x}\frac{\partial}{\partial(x-x')}$ etc.)

Gauss's Law (Differential Version)

$$\vec{\nabla}\cdot\vec{g}(\vec{r}) = \vec{\nabla}\cdot\left(-G\int d^3r'\,\rho(\vec{r}')\,\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}\right)$$

$$= -G\int d^3r'\,\rho(\vec{r}')\,\vec{\nabla}\cdot\left(\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}\right)$$

$$= -G\int d^3r'\,\rho(\vec{r}')\,4\pi\,\delta^{(3)}(\vec{r}-\vec{r}')$$

$$\boxed{\vec{\nabla}\cdot\vec{g}(\vec{r}) = -4\pi G\rho(\vec{r})}$$

Gauss's Law (Integral form)

$$\int_V \vec{\nabla}\cdot\vec{g}\,d^3r \overset{\text{Div}}{\underset{\text{thm.}}{=}} \oint_S \vec{g}\cdot d\vec{A}$$

but $\int_V \vec{\nabla}\cdot\vec{g}\,d^3r = \int_V -4\pi G\rho(\vec{r})\,d^3r = -4\pi G\int_V \rho(\vec{r})\,d^3r = -4\pi G\,M_{enclosed}$

$$\boxed{\Rightarrow \oint_S \vec{g}\cdot d\vec{A} = -4\pi G\,M_{enclosed} \qquad M_{enc} = \int_V \rho(\vec{r})\,d^3r}$$

In problems w/ very symmetric $\rho(\vec{r})$, GL in integral form lets you trivially calculate $\vec{g}(\vec{r})$ (compared to brute force evaluation of $\vec{g} = -G\int \rho(\vec{r}')\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}$)