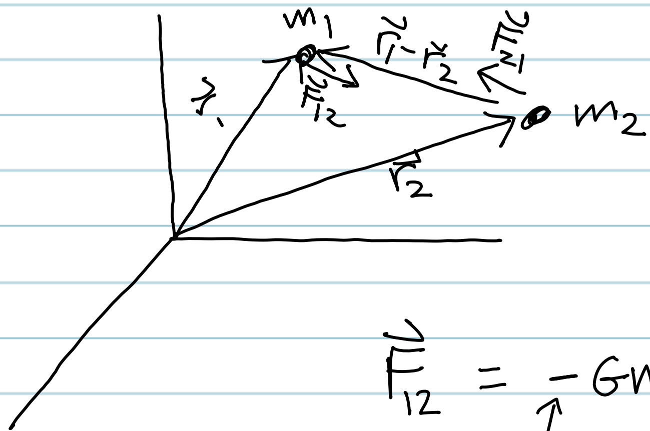
Chapter 5 – Gravitation

Newton's Universal Law of Gravitation:

attractive force between 2 masses



$$\vec{F}_{12} = -Gm_1m_2 \frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|^3}$$

attractive

$$G = 6.673 \times 10^{-11} \frac{N \cdot m^2}{kg^2}$$ "Universal Gravitation Constant"

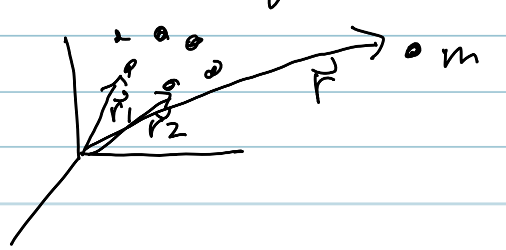
* NOTE on Notation: let $\vec{r} = \vec{r}_1 - \vec{r}_2$

$$\hat{e}_r = \frac{\vec{r}}{r}$$

$$\Rightarrow \boxed{\vec{F}_{12} = \frac{-Gm_1m_2}{r^2}\hat{e}_r}$$

Superposition Principle

* Suppose we have N masses $m_1, m_2, \ldots m_N$.
* What's the force $\vec{F}$ on a mass m at $\vec{r}$

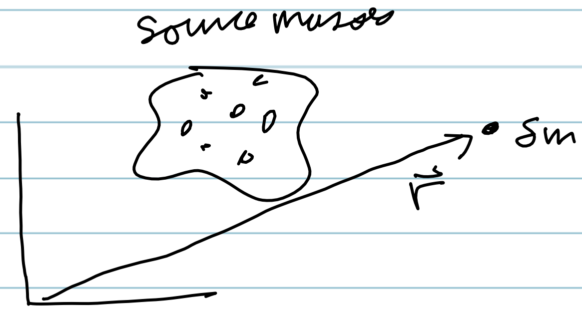


$$\vec{F}(m \text{ at } \vec{r}) = \sum_{i=1}^{N} \frac{-Gmm_i}{|\vec{r} - \vec{r}_i|^3}(\vec{r} - \vec{r}_i)$$

i.e., total force on m = vector sum of forces from the different m_i

Gravitational Field $\vec{g}(\vec{r})$

"test mass" δm in the presence of N point masses m_i i



$$\boxed{g(\vec{r}) \equiv \lim_{\delta m \to 0} \frac{\vec{F}(\delta m \text{ at } \vec{r})}{\delta m}} = \sum_{i=1}^{N} -Gm_i \frac{\vec{r}-\vec{r}_i}{|\vec{r}-\vec{r}_i|^3}$$

NOTE: Why $\delta m \to 0$? Because we dont want the test mass to disturb the arrangement of source masses due to its gravitational forces

Continuous Mass Distribution

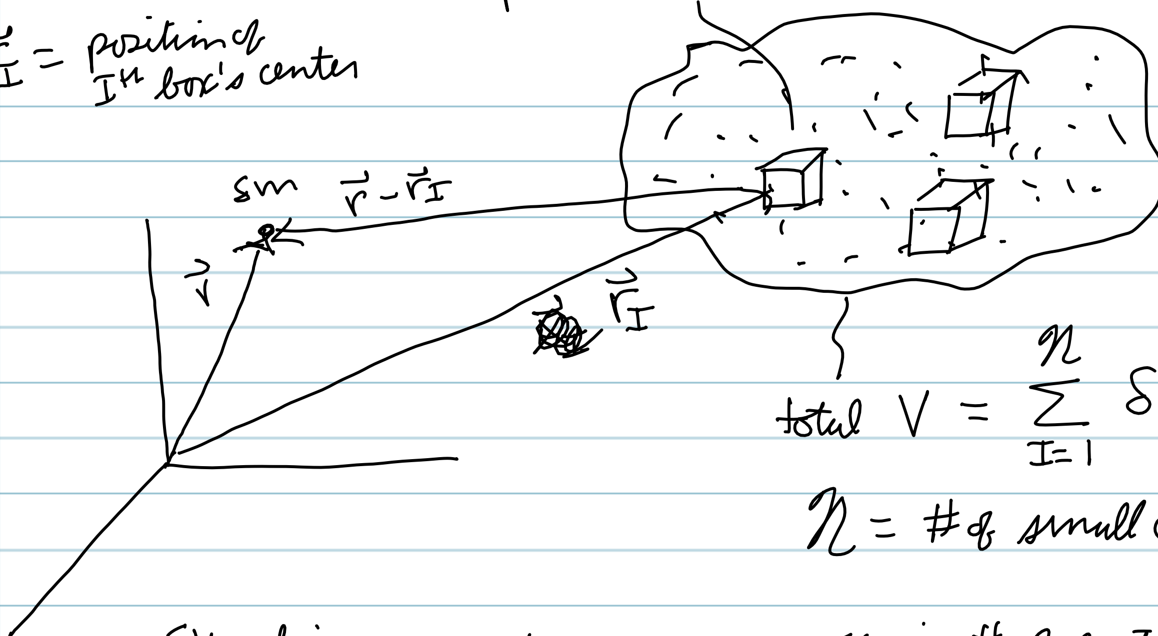
* Consider a macroscopic body of $\sim 10^{23}$ particles
* Makes little sense to work with things like

$$g(\vec{r}) = \sum_{i=1}^{10^{23}} -Gm_i \frac{\vec{r}-\vec{r}_i}{|\vec{r}-\vec{r}_i|^3}$$

* Here we derive the so-called continuum limit

$$g(\vec{r}) = -G \int_{Vol} d^3r' \rho(\vec{r}') \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}$$

$\vec{r}_I$ = position of I^{th} box's center



$$\text{total } V = \sum_{I=1}^{n} \delta V_I$$

n = # of small cubes

δV big enough so macroscopic # of particles in each one

but

δV small enough so $\vec{r}_i \simeq$ constant $= \vec{r}_I$ in δ

$$\Rightarrow \quad g(\vec{r}) = \sum_{i=1}^{10^{23}} \frac{-Gm_i(\vec{r}-\vec{r}_i)}{|\vec{r}-\vec{r}_i|^3}$$

$$= \sum_{I=1}^{n} \sum_{j\in\delta V_I} \frac{-Gm_j}{|\vec{r}-\vec{r}_j|^3}(\vec{r}-\vec{r}_j)$$

$$\simeq \sum_{I=1}^{n} -G \frac{\vec{r}-\vec{r}_I}{|\vec{r}-\vec{r}_I|^3} \sum_{j\in\delta V_I} m_j$$

(since $\vec{r}_j \simeq \vec{r}_I$ for all j in I^{th} bo

<u>Mass Df. Density</u>:

$$\boxed{\rho(\vec{r}_I)\,\delta V_I = \sum_{j\in\delta V_I} m_j}$$

$$\Rightarrow \quad g(\vec{r}) \simeq \sum_{I=1}^{n} -G\,\rho(\vec{r}_I)\,\delta V_I \frac{\vec{r}-\vec{r}_I}{|\vec{r}-\vec{r}_I|^3}$$

$$\overset{\delta V\to 0}{=} -G\int d^3r'\,\rho(\vec{r}\,')\frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}$$

So, $\boxed{\vec{g}(\vec{r}) = -G \int_V d^3r' \, \rho(\vec{r}') \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}}$ ⊗

Beware of annoying (misleading) notation in the book.

$$\vec{g} = -G \int \frac{\rho(\vec{r}')}{\mathscr{r}^2} \hat{e}_{\mathscr{r}} \, d^3r'$$

$$\vec{\mathscr{r}} = \vec{r} - \vec{r}'$$

Generalization of ⊗ to 2d (surface) + 1d (line) distribution

$\rho(\vec{r}')d^3r' = dm$ in d^3r' about $\vec{r}'$

$\sigma(\vec{r}')dA' = dm$ in dA' about $\vec{r}'$

$\lambda(\vec{r}')dl' = dm$ in dl' about $\vec{r}'$

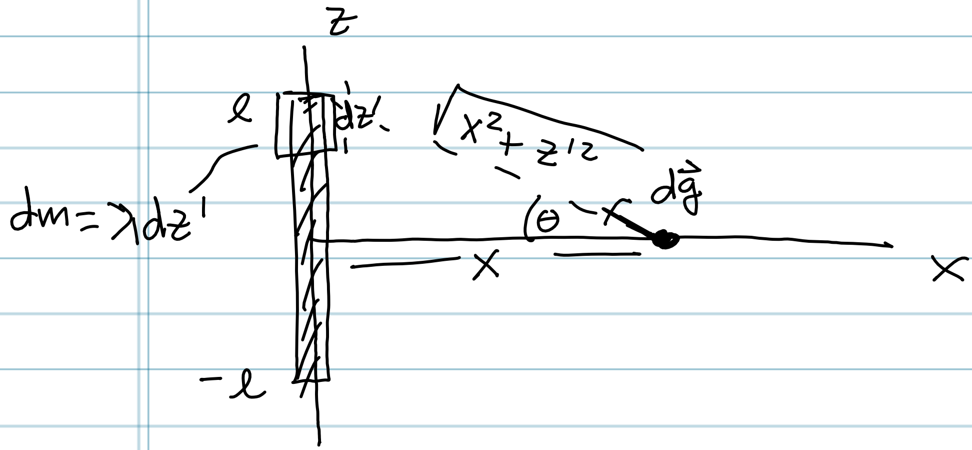
$\Rightarrow$

$$\vec{g}(\vec{r}) = -G \int_{\text{Surface}} dA' \, \sigma(\vec{r}') \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}$$ 2D mass distribution

$$\vec{g}(\vec{r}) = -G \int_{\text{Line}} dl' \, \lambda(\vec{r}') \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}$$ Linear ma[ss] distribution

Example: Uniform wire $\lambda(\vec{r}\,') = \lambda$ on z-axis

* Find $\vec{g}(x,0,0)$



By symmetry, $\vec{g}(x,0,0) = g_x(x,0,0)\,\hat{\imath}$

$$\therefore \; g_x(x,0,0) = \int dg_x(x,0,0)$$

$$= \int d\vec{g}(x,0,0)\cdot\hat{\imath}$$

but $dg_x(x,0,0) = -G\,dm \dfrac{\cos\theta}{x^2+z'^2}$

$$\cos\theta = \frac{x}{\sqrt{x^2+z'^2}}$$

$$\Rightarrow dg_x(x,0,0) = -G\lambda\,dz' \frac{x}{(x^2+z'^2)^{3/2}}$$

$$\boxed{g_x(x,0,0) = -G\lambda x \int_{-l}^{l} \frac{dz'}{(x^2+z'^2)^{3/2}}}$$

Integral is elementary (OK, here I cheated & plugged into mathematica)

$$\int_{-\ell}^{\ell} \frac{dz'}{(x^2+z'^2)^{3/2}} = \frac{2\ell}{x^2\sqrt{\ell^2+x^2}}$$

$$\Rightarrow \boxed{g(x,0,0) = -G\lambda x \frac{2\ell}{x^2\sqrt{\ell^2+x^2}}}$$

<u>Does it make sense? Check an 'easy' limit</u>

for $x \gg \ell$, the line mass should resemble a point particle of mass $m = (2\ell\lambda)$

$$g_x(x,0,0) = \frac{-Gmx}{x^2\sqrt{\ell^2+x^2}} = \frac{-Gm\cancel{x}}{x^3\sqrt{1+\ell^2/x^2}}$$

$$\approx -\frac{Gm}{x^2}\left(1 + O\left(\frac{\ell^2}{x^2}\right)\right)$$