Reminders

① Exam 2 on Friday 3/15

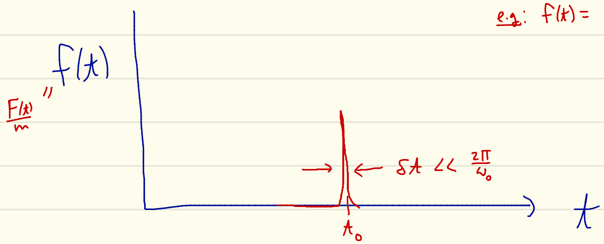
- inelastic collision
- cross section
- rockets
- ch. 3 (oscillation thru end of today's class)

② Pls. start reading Ch 4 (Non-linear Osc.)

③ I'll post a practice exam on the webpage (where lecture notes are)
+ we'll go over it after break.

Recap of Impulsive driving forces + Green's Function Method

* What happens if a damped HO at rest is given a kick (impulse) at t_0 that lasts a time $\delta t \to 0$?



e.g.: $f(t) = \begin{cases} 0 & t < t_0 \\ \frac{v_0}{\delta t} & \text{if} \quad t_0 \le t \le t_0 + \delta t \\ 0 & \text{if} \quad t > t_0 + \delta t \end{cases}$

- after kick acts there's momentum $m v_0 = \int_{t_0}^{t_0+\delta t} F\,dt \simeq F(t_0)\delta t \Rightarrow v_0 = \frac{F(t_0)}{m}\delta t \equiv f(t_0)\delta t$

- No driving force after the kick is done.

$\Downarrow$

Solution for damped HO: $\quad X(t>t_0) = A e^{-\beta t}\cos(\omega_1 t + \delta)$

w/ IC's $\quad$ (1) $X(t_0) = 0$

(2) $\dot{X}(t_0) = v_0$

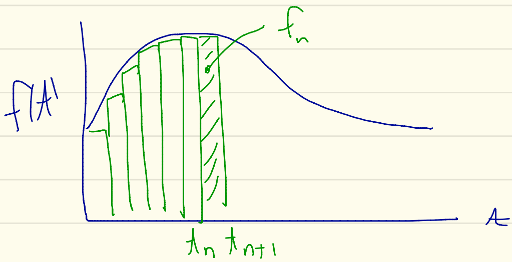
$\Rightarrow$ $\boxed{X(t) = \begin{cases} 0 & t < X_0 \\ \frac{v_0}{\omega_1} e^{-\beta(t-t_0)} \sin[\omega_1(t-t_0)] & t \ge X_0 \end{cases}}$

Green's Function Method

How to solve $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f(t)$

for arbitrary $f(t)$?

Idea: treat $f(t) = \sum_n f_n$ = sum of impulsive forces & use $x = \sum_n x_n$



$\Rightarrow$ In limit $\delta t \to 0$ between neighboring impulses,

$$\boxed{x(t) = \int_{-\infty}^{\infty} G(t,t')\, f(t')\, dt'}$$

$$G(t,t') \equiv \begin{cases} 0 & \text{if } t' > t \\ \dfrac{e^{-\beta(t-t')}}{\omega_1}\sin[\omega_1(t-t')] & \text{if } t' \le t \end{cases}$$

* Sidenote for those familiar w/ Dirac Delta function

$$\hat{L}_t G(t,t') = \delta(t-t') \Rightarrow \hat{L}_t x(t) = \hat{L}_t \int G(t,t') f(t')\, dt' = \int \overbrace{\left(\hat{L}_t G(t,t')\right)}^{\delta(t-t')} f(t')\, dt' = f(t) \quad \checkmark$$

Ex: find $X(t)$ for $f(t) = \begin{cases} 0 & t<0 \\ f_0 & t \geq 0 \end{cases}$

soln: $X(t) = \int_{-\infty}^{\infty} G(t,t') f(t')\, dt' = \int_{-\infty}^{t} \frac{e^{-\beta(t-t')}}{\omega_1} \sin[\omega_1(t-t')]\, f(t')\, dt'$

$$= \frac{f_0}{\omega_1} \int_0^t e^{-\beta(t-t')} \sin[\omega_1(t-t')]\, dt'$$

use $e^{i\theta} = \cos\theta + i\sin\theta \Rightarrow \sin\theta = \text{Im}[e^{i\theta}]$

$$\therefore \int_0^t e^{-\beta(t-t')} \sin[\omega_1(t-t')] = \text{Im}\Big[\underbrace{\int_0^t e^{-\beta(t-t')} \times e^{i\omega_1(t-t')}\, dt'}_{ⓐ} \Big]$$

$$ⓐ = \int_0^t e^{(t-t')[i\omega_1 - \beta]}\, dt'$$

* let $t - t' = \tau$
$dt' = -d\tau$

$$= \int_0^t e^{\tau[i\omega_1-\beta]}\, d\tau = \frac{1}{[i\omega_1-\beta]} e^{\tau[i\omega_1-\beta]} \Big|_0^t = \frac{1}{\beta - i\omega_1}\left(1 - e^{\tau(i\omega_1-\beta)}\right)$$

$$= \frac{\beta + i\omega_1}{\beta^2+\omega_1^2}\left(1 - e^{i\omega_1 t} e^{-\beta t}\right)$$

$$ⓐ = \frac{\beta+i\omega_1}{\beta^2+\omega_1^2}\left[1 - \cos\omega_1 t\, e^{-\beta t} - i\sin\omega_1 t\, e^{-\beta t}\right]$$

$$\text{Im}[ⓐ] = -\frac{\omega_1 e^{-\beta t}}{\beta^2+\omega_1^2}\cos\omega_1 t - \frac{\beta e^{-\beta t}\sin\omega_1 t}{\beta^2+\omega_1^2} + \frac{\omega_1}{\beta^2+\omega_1^2}$$

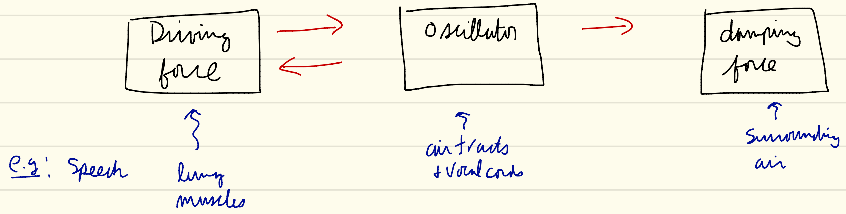
* but $\beta^2 + \omega_1^2 = \omega_0^2$

$$= -\frac{\omega_1}{\omega_0^2} e^{-\beta t}\cos\omega_1 t - \frac{\beta}{\omega_0^2} e^{-\beta t}\sin\omega_1 t + \frac{\omega_1}{\omega_0^2}$$

$$\boxed{\begin{aligned} \Rightarrow X(t) &= \frac{f_0}{\omega_0^2}\left[1 - e^{-\beta t}\cos\omega_1 t - \frac{\beta}{\omega_1} e^{-\beta t}\sin\omega_1 t\right] \quad \text{for } t>0 \\ &= 0 \quad t<0. \end{aligned}}$$

Energy transfer thru a driven Oscillator

* energy transfer through Oscillator important for many processes



* Work done by a force: $dW = \vec{F}\cdot d\vec{r}$

* Instantaneous power delivered by Force: $P = \frac{dW}{dt} = \vec{F}\cdot\frac{d\vec{r}}{dt} = \vec{F}\cdot\vec{v}$

$F_{damp} = -b\dot{x} \Rightarrow P_{damp} = -b\dot{x}^2 \leq 0$ takes energy away from oscillator (e.g.: heat)

$= -b^2 D^2 \omega^2 \sin^2(\omega t - \delta)$ $\beta = \frac{b}{2m}$ $b = 2m\beta$

$\Rightarrow \langle P_{damp} \rangle = -\frac{1}{2} b^2 D^2 \omega^2 = -2m^2\beta^2 D^2(\omega)\,\omega^2$

$F_{drive} = F_0 \cos\omega t \Rightarrow P_{drive} = -F_0\cos\omega t \cdot \omega D \sin(\omega t - \delta) = -\omega D F_0 \cos\omega t \sin(\omega t - \delta)$

$\delta = \tan^{-1}\left(\frac{2\omega\beta}{\omega_0^2 - \omega^2}\right) \Rightarrow$ 1) $\omega \to 0,\ \delta \to 0 \Rightarrow P_{drive} = -\omega D F_0 \cos\omega t \sin\omega t$ (> or < 0)

2) $\omega \to \infty,\ \delta \to \pi \Rightarrow P_{drive} = \omega D F_0 \cos\omega t \sin\omega t$ (> or < 0)

3) $\omega \to \omega_0$ $\delta \to \frac{\pi}{2}$ $P_{drive} = \omega D F_0 \cos^2\omega t$ (> 0 only)

$\delta = \tan^{-1}\left(\frac{2\omega\beta}{\omega_0^2 - \omega^2}\right) \Rightarrow$ 1) $\omega \to 0,\ \delta \to 0 \Rightarrow P_{drive} = -\omega D F_0 \cos\omega t \sin\omega t$ (> or < 0)

2) $\omega \to \infty,\ \delta \to \pi \Rightarrow P_{drive} = \omega D F_0 \cos\omega t \sin\omega t$ (> <u>or</u> < 0)

3) $\omega \to \omega_0\quad \delta \to \frac{\pi}{2}\quad P_{drive} = \omega D F_0 \cos^2\omega t$ (> 0 <u>only</u>)

1) & 2) $\Rightarrow$ energy sloshes back & forth between oscillator & driving force.

However, $\langle P_{drive} \rangle = 0$

3) $\Rightarrow$ No sloshing back & forth. Energy strictly delivered from driving term to oscillator

$$\langle P_{drive} \rangle = \frac{\omega D F_0}{2}$$