Differential Vector Calculus Review

I) Gradient

1d: How does $f(x)$ vary if we change $x \longrightarrow x+dx$

ans: $df(x) = f(x+dx) - f(x)$

$$= \cancel{f(x)} + \frac{\partial f}{\partial x}dx - \cancel{f(x)} + \mathcal{O}(dx^2)$$

$$\boxed{\therefore\ df(x) = \frac{df}{dx}dx}$$

3d generalization: $f(\vec{x}) \equiv f(x,y,z)$ (or $f(x_1,x_2,x_3)$)

$$df(\vec{x}) = f(\vec{x}+d\vec{x}) - f(\vec{x})$$

$$= f(x_1+dx_1, x_2+dx_2, x_3+dx_3) - f(x_1,x_2,x_3)$$

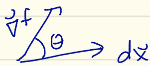
$$= \cancel{f(x_1,x_2,x_3)} + \sum_{i=1}^{3} \frac{\partial f}{\partial x_i}dx_i + \mathcal{O}(d\vec{x}^2) - \cancel{f(x_1,x_2,x_3)}$$

$$\therefore\ df(\vec{x}) = \sum_i \frac{\partial f}{\partial x_i}dx_i$$

Looks like a dot product!

* Define: $\boxed{\vec{\nabla} f = \sum_{i=1}^{3} \frac{\partial f}{\partial x_i}\hat{e}_i}$ (sometimes written "grad f")

$$\Rightarrow \boxed{\begin{aligned} df(\vec{x}) &= \vec{\nabla} f \cdot d\vec{x} = |\vec{\nabla} f||d\vec{x}|\cos\theta \\ &= \sum_i \frac{\partial f}{\partial x_i}dx_i \quad \text{(cartesian coordinates only!)} \end{aligned}}$$



* Note: $d\vec{x} \equiv dx_1\hat{e}_1 + dx_2\hat{e}_2 + dx_3\hat{e}_3$

Clicker Question # 1 Find $|\vec{\nabla}\Phi|$ at $\vec{x} = (0,0,0)$

$$\Phi(\vec{x}) = 12 - 2x + xy + y^2 - z^2$$

(A) -1.0

(B) 0.0

(C) 2.0

(D) 4.0

(E) 6.0

Clicker ANSWER # 1

Find $|\vec{\nabla}\Phi|$ at $\vec{x} = (0,0,0)$

$$\Phi(\vec{x}) = 12 - 2x + xy + y^2 - z^2$$

(A) -1.0

(B) 0.0

(C) 2.0

(D) 4.0

(E) 6.0

at $(0,0,0)$

$$\frac{\partial \Phi}{\partial x} = -2 + y = -2$$

$$\frac{\partial \Phi}{\partial y} = x + 2y = 0$$

$$\frac{\partial \Phi}{\partial z} = -2z = 0$$

$$\Downarrow$$

$$|\vec{\nabla}\Phi| = \sqrt{(-2)^2 + (0)^2 + (0)^2}$$

$$= 2.0$$

* Quick aside on Vector functions (aka vector fields)

Vector function (field) $\vec{F}(\vec{x})$ (sometimes written as $\vec{F}(x,y,z)$ etc.)

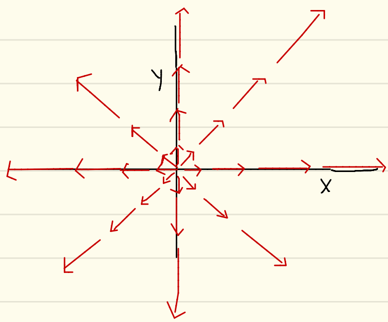
$\Rightarrow$ to each point in space $\vec{x}$, we assign a vector

Examples: electric field $\vec{E}(\vec{x})$, fluid velocity profile $\vec{v}(\vec{x})$, etc.

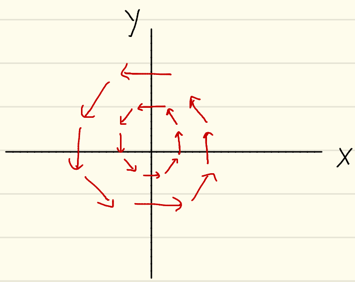
$\vec{\nabla} f(\vec{x})$ is a vector function.

* Graphing a vector field:

Ex 1: $\vec{F}(\vec{x}) = \vec{x} = x\hat{i} + y\hat{j}$



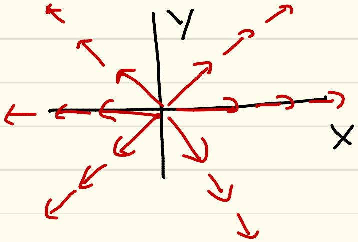
Ex 2: $\vec{F}(\vec{x}) = -y\,\hat{i} + x\,\hat{j}$



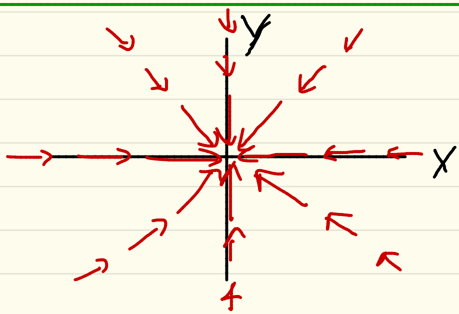
* Clicker question #2: Let $\vec{F}(x,y) = x\,\hat{j}$

Which plot of $\vec{F}$ is correct?

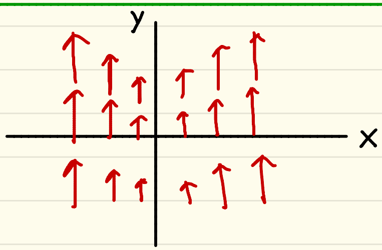
A.)



B.)



C.)

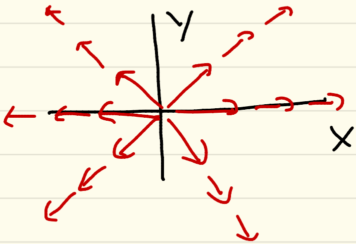


D.)

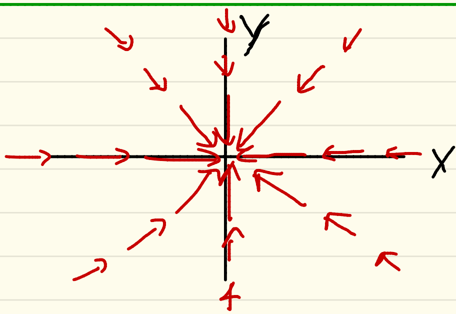
*** ANSWER #2:** Let $\vec{F}(x,y) = x\hat{j}$

Which plot of $\vec{F}$ is correct?

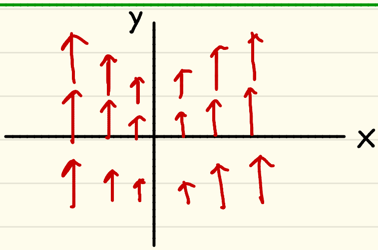
A)



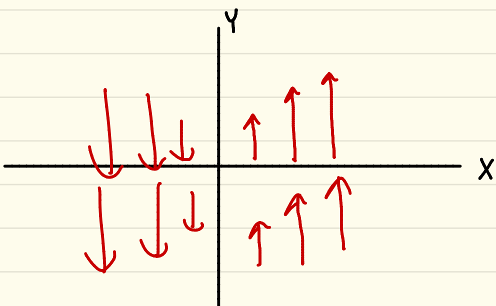
B)



C)

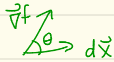


D)



* Physical Meaning of $\vec{\nabla} f$

$$df = \vec{\nabla} f \cdot d\vec{x} = |\vec{\nabla} f|\,|d\vec{x}| \cos\theta$$

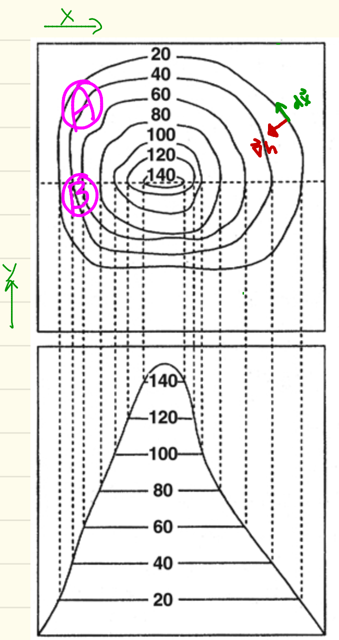


* For fixed $|d\vec{x}|$, vary θ to see in what direction is df maximal

$\Rightarrow \cos\theta = 1 \Rightarrow d\vec{x} \parallel \vec{\nabla} f$ gives biggest df

> 1) $\vec{\nabla} f(\vec{x})$ points in direction of steepest increase of $f(\vec{x})$ at $\vec{x}$.
>
> 2) $|\vec{\nabla} f(\vec{x})|$ = rate of steepest increase at $\vec{x}$ (i.e., slope)

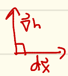
Example: $h(x,y)$ = elevation on a mountain



What direction should you step at a given point on the mountain to stay @ constant elevation?

ans: Want $dh(x,y) = |\vec{\nabla} h|\,|d\vec{x}| \cos\theta = 0$

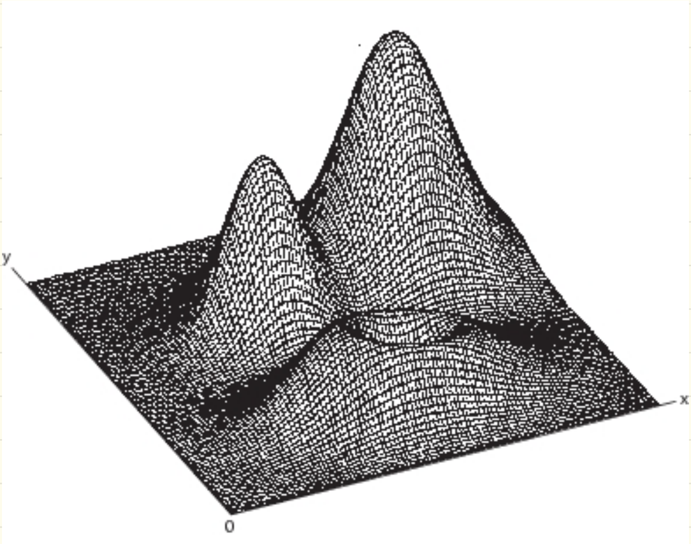
$\therefore \theta = \frac{\pi}{2} = 90°$

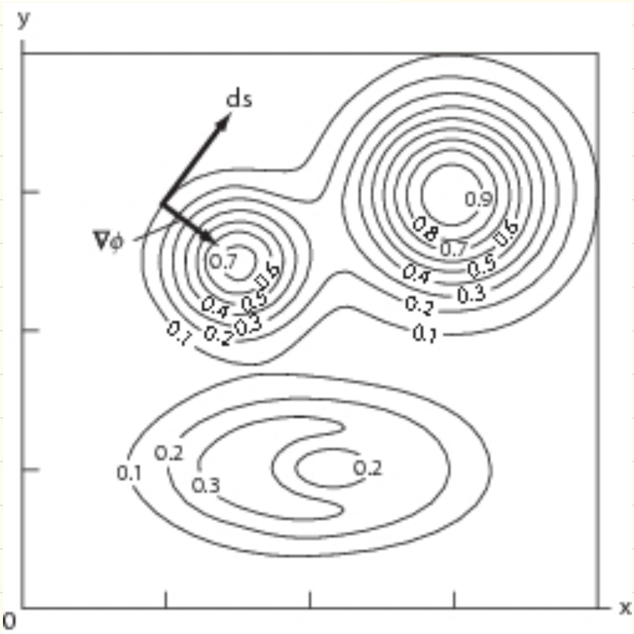


*** where is $|\vec{\nabla} h|$ bigger, A or B?**

ans: at B (contours closer together)

A slightly nicer picture





* A useful special case to remember

* Suppose $f(x,y,z) = f(r)$ where $r = |\vec{x}| = \sqrt{x^2+y^2+z^2}$

$$\vec{\nabla} f(r) = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} + \frac{\partial f}{\partial z}\hat{k}$$

*but $\frac{\partial f}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial f}{\partial r}$, $\frac{\partial f}{\partial y} = \frac{\partial r}{\partial y}\frac{\partial f}{\partial r}$, $\frac{\partial f}{\partial z} = \frac{\partial r}{\partial z}\frac{\partial f}{\partial r}$

$$\frac{\partial r}{\partial x} = \frac{1}{\cancel{2}\sqrt{x^2+y^2+z^2}}\cancel{2}x = \frac{x}{r}$$ (& likewise for $\frac{\partial r}{\partial y}$ & $\frac{\partial r}{\partial z}$)

$$\therefore \vec{\nabla} f(r) = \left(\frac{x}{r}\hat{\imath} + \frac{y}{r}\hat{\jmath} + \frac{z}{r}\hat{k}\right)\frac{df}{dr}$$

$$= \frac{\vec{x}}{r}\frac{df}{dr}$$ * df. $\hat{r} \equiv \frac{\vec{x}}{r} = \frac{\vec{x}}{|\vec{x}|}$

$$\Downarrow$$

$$\boxed{\vec{\nabla} f(r) = \hat{r}\frac{df}{dr}}$$