Reminders

1) HW 6 due this friday (1st 2 problems were essentially done in class last week).

2) No office hr tomorrow (Tues). Makeup Wednesday from 11-12.

3) Exam Friday 3/15

incl. collision
cross section
rocket motion
oscillation (ch. 3).

Ex: Energy loss of lightly damped oscillator (# 3.11)

$$E(t) = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} k x^2 = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m \omega^2 x^2$$

$$x = A e^{-\beta t} \cos(\omega_1 t - \delta) \qquad \omega_1 = \sqrt{\omega_0^2 - \beta^2}$$

$$\dot{x} = A e^{-\beta t} \left[-\beta \cos(\omega_1 t - \delta) - \omega_1 \sin(\omega_1 t - \delta) \right]$$

$$\Downarrow$$

$$E(t) = \frac{A^2}{2} e^{-2\beta t} \left[(m\beta^2 + k) \cos^2(\omega_1 t - \delta) + m\omega_1^2 \sin^2(\omega_1 t - \delta) + 2m\beta\omega_1 \sin(\omega_1 t - \delta) \cos(\omega_1 t - \delta) \right]$$

$$= \frac{mA^2}{2} e^{-2\beta t} \left[\beta^2 \cos 2(\omega_1 t - \delta) + \beta \sqrt{\omega_0^2 - \beta^2} \sin 2(\omega_1 t - \delta) + \omega_0^2 \right]$$

$$\Downarrow$$

$$\frac{dE}{dt} = \frac{mA^2}{2} e^{-2\beta t} \left[(2\beta\omega_0^2 - 4\beta^3) \cos 2(\omega_1 t - \delta) - 4\beta^2 \sqrt{\omega_0^2 - \beta^2} \sin 2(\omega_1 t - \delta) - 2\beta\omega^2 \right]$$ // general result

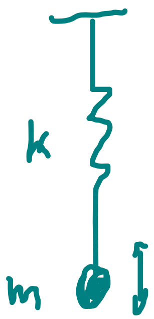
* Now let's make 2 approx

1) $\beta \ll \omega_0$
2) take $\left\langle \frac{dE}{dt} \right\rangle = \frac{1}{T} \int_0^T \frac{dE}{dt} \, dt$ $\qquad T = \frac{2\pi}{\omega_1} \simeq \frac{2\pi}{\omega_0}$

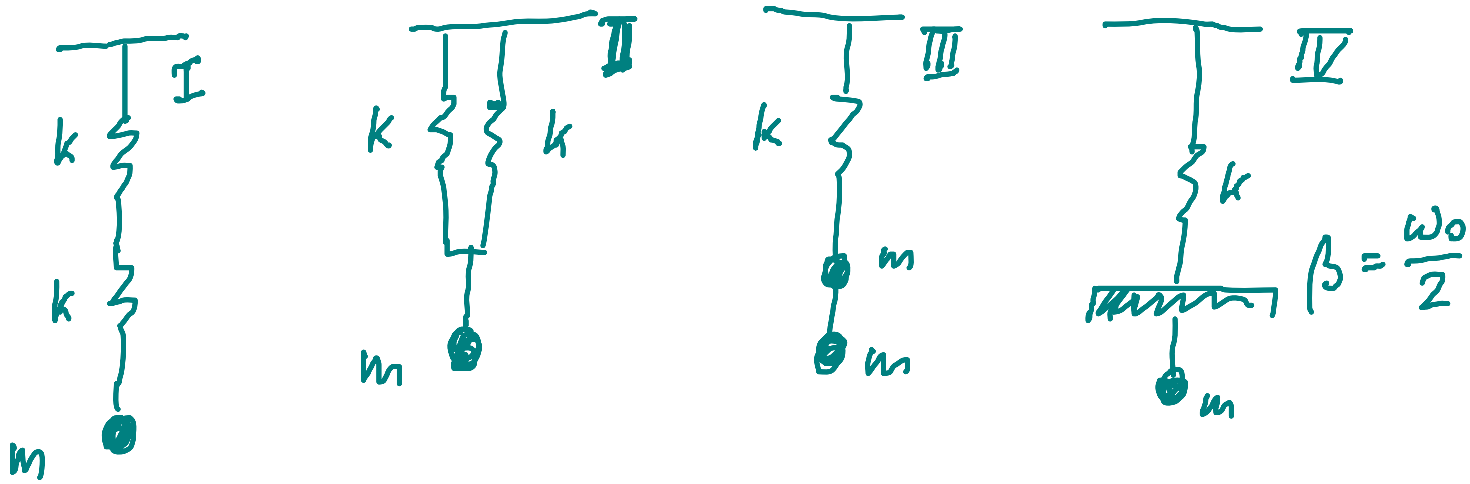
$\Rightarrow$ $e^{-2\beta t} \approx$ const over T

$$\langle \cos 2(\omega_1 t - \delta) \rangle \simeq \langle \sin 2(\omega_1 t - \delta) \rangle = 0$$

$$\boxed{\therefore \left\langle \frac{dE}{dt} \right\rangle_{\beta \ll \omega_0} \simeq -m\beta\omega_0^2 A^2 e^{-2\beta t}}$$



MASS m ATTACHED TO A SPRING OF SPRING CONSTANT k OSCILLATES AT AN ANGULAR FREQUENCY $\omega_0 = \sqrt{\frac{k}{m}}$. WHICH CHANGES TO THE SYSTEM WILL RESULT IN AN INCREASE OF THE PERIOD OF OSCILLATIONS BY $\sqrt{2}$?



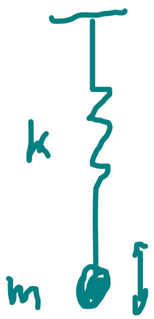
A) II

B) I, III, IV

C) II, III

D) II, IV

E) I, III



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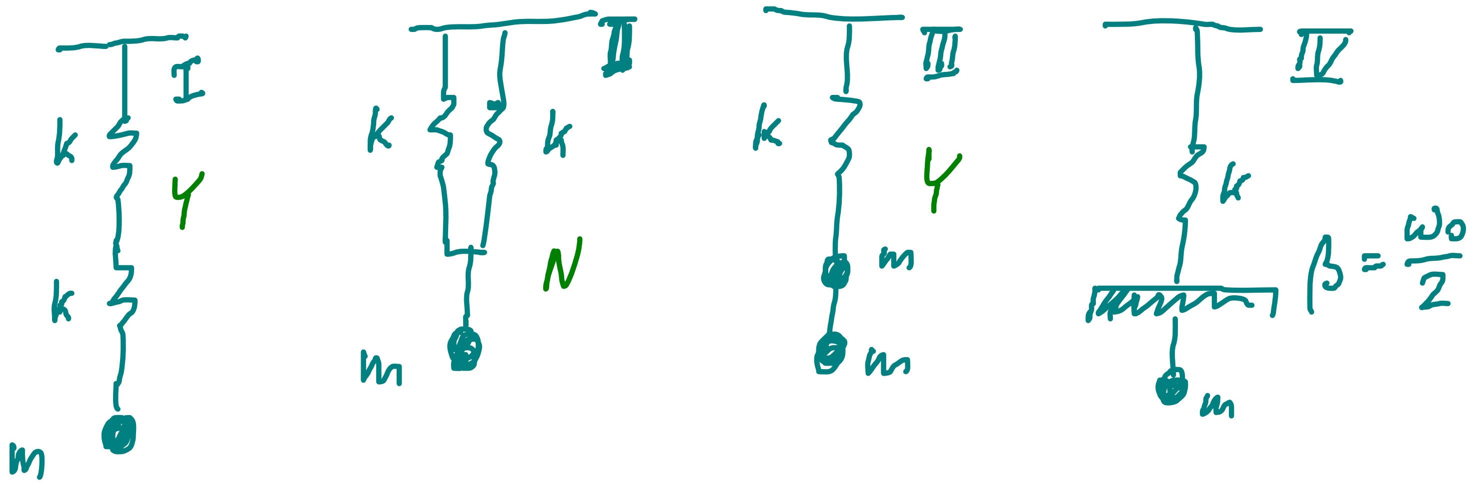


A) II

B) I, III, IV

C) II, III

D) II, IV

E) I, III

$$k \to \frac{k}{2}$$

$$\sqrt{\frac{k}{2m}} \qquad T \propto \sqrt{\frac{2m}{k}}$$

$$\omega_1^2 = \sqrt{\omega_0^2 - \beta^2}$$

$$= \sqrt{\omega_0^2 - \frac{1}{4}\omega_0^2}$$

$$= \omega_0 \frac{\sqrt{3}}{2}$$

* Damped HO w/ periodic driving forces

$$F = -kx - b\dot{x} + \underbrace{F_0 \cos\omega t}_{\text{driving force to keep the system oscillating}}$$

(generally, $\omega \neq \omega_0$)

$$\therefore \; m\ddot{x} = -kx - b\dot{x} + F_0 \cos\omega t$$

$$\Rightarrow \boxed{\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f\cos\omega t}$$

(ω_0 + β as before, $f \equiv \frac{F_0}{m}$)

Inhomogeneous 2^{nd}-order ODE. Read App. C !!

General way to solve such equations:

$$x(t) = x_{hg}(t) + x_p(t)$$

Sol'n to the Homogeneous eqn. (i.e., w/out driving term)

$$x_{hg} = e^{-\beta t}\left(A_1 e^{\sqrt{\beta^2-\omega_0^2}\,t} + A_2 e^{-\sqrt{\beta^2-\omega_0^2}\,t}\right)$$

$x_p(t)$ = "particular sol'n" that obeys

$$\ddot{x}_p + 2\beta\dot{x}_p + \omega_0^2 x_p = f\cos\omega t$$

let $\hat{L} = \frac{d^2}{dt^2} + 2\beta \frac{d}{dt} + \omega_0^2$ ("linear operator")

$\Rightarrow \hat{L}(x_{hg} + x_p) = \underbrace{\hat{L} x_{hg}}_{0} + \hat{L} x_p = f \cos \omega t$

$\Rightarrow x = x_{hg} + x_p$ is indeed a soln & x_{hg} has 2 constants to match $x(0), \dot{x}(0)$

Guess: $x_p(t) = D \cos(\omega t - \delta)$

$\dot{x}_p = -\omega D \sin(\omega t - \delta)$

$\ddot{x}_p = -\omega^2 D \cos(\omega t - \delta) = -\omega^2 x_p$

$\hat{L} x_p = (\omega_0^2 - \omega^2) D \cos(\omega t - \delta) - 2\beta\omega D \sin(\omega t - \delta) = f \cos \omega t$

* $\cos(\omega t - \delta) = \cos \omega t \cos \delta + \sin \omega t \sin \delta$

* $\sin(\omega t - \delta) = \sin \omega t \cos \delta - \cos \omega t \sin \delta$

$\Rightarrow (\omega_0^2 - \omega^2) D [\cos \omega t \cos \delta + \sin \omega t \sin \delta] - 2\beta\omega D [\sin \omega t \cos \delta - \cos \omega t \sin \delta] = f \cos \omega t$

$\Downarrow$ re-group $\cos \omega t$ & $\sin \omega t$ terms

$\cos \omega t \{ \underline{(\omega_0^2 - \omega^2) D \cos \delta + 2\beta\omega D \sin \delta - f} \} + \sin \omega t \{ \underline{(\omega_0^2 - \omega^2) D \sin \delta - 2\beta\omega D \cos \delta} \} = 0$

*Key Point: $\sin \omega t$ & $\cos \omega t$ are linearly independent functions

$A \cos \omega t + B \sin \omega t = 0$

only possible if $A = B = 0$

$$\cos\omega t\left\{\underline{(\omega_0^2-\omega^2)D\cos\delta + 2\beta\omega D\sin\delta - f}\right\} + \sin\omega t\left\{\underline{(\omega_0^2-\omega^2)D\sin\delta - 2\beta\omega D\cos\delta}\right\} = 0$$

$$\Downarrow$$

$$(\omega_0^2-\omega^2)D\sin\delta - 2\beta\omega D\cos\delta = 0 \quad (1)$$

2 eqs 2 unknowns (D, δ)

$$(\omega_0^2-\omega^2)D\cos\delta + 2\beta\omega D\sin\delta - f = 0 \quad (2)$$

eq.(1) $\Rightarrow$ $$\boxed{\tan\delta = \frac{2\beta\omega}{\omega_0^2-\omega^2}} \quad (*)$$

(2) $\Rightarrow$ $$f^2 = \left[(\omega_0^2-\omega^2)^2\cos^2\delta + 4\beta^2\omega^2\sin^2\delta + 4\beta\omega(\omega_0^2-\omega^2)\sin\delta\cos\delta\right]D^2$$

* but eq. (*) $\Rightarrow$ $$\sin\delta = \frac{2\omega\beta}{\sqrt{(\omega_0^2-\omega^2)^2+4\omega^2\beta^2}}$$

$$\cos\delta = \frac{\omega_0^2-\omega^2}{\sqrt{(\omega_0^2-\omega^2)^2+4\omega^2\beta^2}}$$

plug into f^2 & simplify:

$$f^2 = D^2\left[(\omega_0^2-\omega^2)^2 + 4\beta^2\omega^2\right]$$

$$\Rightarrow \boxed{D = \frac{f}{\sqrt{(\omega_0^2-\omega^2)^2+4\beta^2\omega^2}}} \quad (**)$$

Full Solution

$$\boxed{X(t) = e^{-\beta t}\left(A_1 e^{\sqrt{\beta^2-\omega_0^2}\,t} + A_2 e^{-\sqrt{\beta^2-\omega_0^2}\,t}\right) + \frac{f}{\sqrt{(\omega_0^2-\omega^2)^2+4\beta^2\omega^2}}\cos(\omega t-\delta) \qquad \delta = \text{Arctan}\left(\frac{2\beta\omega}{\omega_0^2-\omega^2}\right)}$$

Note: $X_{HG}(t) \xrightarrow[t\to\infty]{} 0$ due to $e^{-\beta t}$ (transient)

$X_p(t)$ remains (steady state sol'n)

Resonance Phenomena

$$D(\omega) = \frac{f}{\sqrt{(\omega_0^2-\omega^2)^2+4\beta^2\omega^2}}$$

what ω maximizes D?

$$\left.\frac{dD}{d\omega}\right|_{\omega=\omega_R} = 0 = -\frac{1}{2\left[(\omega_0^2-\omega_R^2)^2+4\beta^2\omega_R^2\right]^{3/2}} \times \left(2(\omega_0^2-\omega_R^2)\cdot(-2\omega_R) + 8\beta^2\omega_R\right)$$

$$0 = -4\omega_0^2 + 4\omega_R^2 + 8\beta^2$$

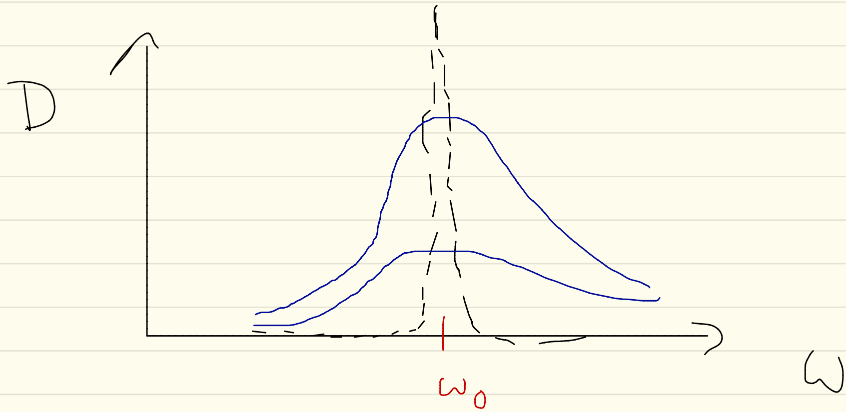
$$\Rightarrow \boxed{\omega_R = \sqrt{\omega_0^2 - 2\beta^2}}$$

*Note: If $\omega_0^2 \leq 2\beta^2$, the No resonance (Why?)

- ω_R imaginary
- No resonance (pure decay)

Quality Factor

$$Q \equiv \frac{\omega_R}{2\beta}$$ characterise degree of damping



Ex: show $Q \simeq 2\pi\left(\frac{E}{\Delta E}\right)$

E = total energy

ΔE = energy loss over 1 period

for driven, lightly damped ($\omega_0 \gg \beta$) system near resonance

Sol'n: $\omega_R = \sqrt{\omega_0^2 - 2\beta^2} \approx \omega_0 \approx \omega$

$\therefore Q \equiv \frac{\omega_R}{2\beta} \approx \frac{\omega_0}{2\beta}$

to get total energy, we ignore transients and use

$$x(t) \simeq D\cos(\omega t - \delta) \Rightarrow \dot{x} = -\omega D \sin(\omega t - \delta)$$

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\omega_0^2 x^2$$

$$= \frac{mD^2}{2}\left[\omega^2\sin^2(\omega t - \delta) + \omega_0^2\cos^2(\omega t - \delta)\right]$$

$$\boxed{E \approx \frac{mD^2\omega_0^2}{2}\Big[\underbrace{\sin^2() + \cos^2()}_{1}\Big]}$$

ΔE = -work done by damping force over 1 period

$$\Delta E = \int F_{damp}\,dx = \int(+b\dot{x})\,dx = \int_0^T (2m\beta\dot{x})\,\dot{x}\,dt = 2\pi m\omega\beta D^2 \qquad \left(T = \frac{2\pi}{\omega}\right)$$

$$\approx 2\pi m\omega_0\beta D^2$$

$$\Rightarrow \frac{E}{\Delta E} = \frac{\frac{1}{2}m\omega_0^2 D^2}{2\pi m\omega_0\beta D^2} = \frac{\omega_0}{4\pi\beta} = \frac{Q}{2\pi}$$

$$\Rightarrow Q \simeq 2\pi\left(\frac{E}{\Delta E}\right) \checkmark$$