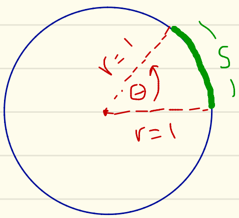
* A quick review on Solid Angles

Solid angle Ω - used to quantify a range of directions in 3D

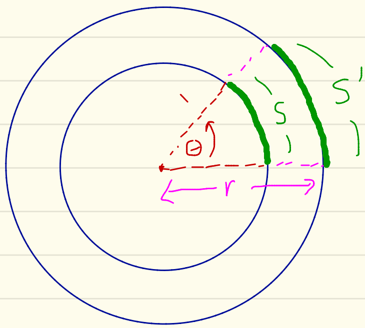
* Take a step back & recall what an angle θ means in 2D.

Consider a unit $(r=1)$ circle



$\theta = S$

* more generally, for a circle of radius $r \neq 1$

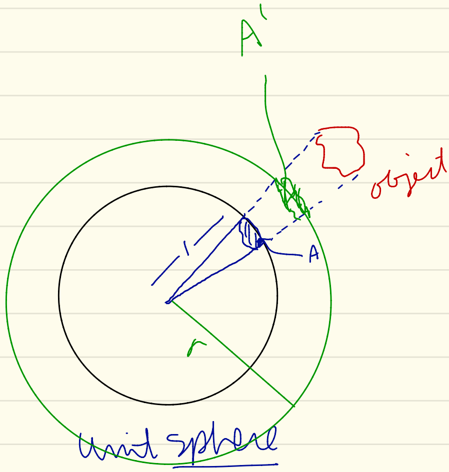


$\theta = S = \dfrac{S'}{r}$
(radians)

$\Downarrow$

$\theta_{max} = 2\pi$

Solid Angle: Quantifies range of directions in 3D, e.g., for seeing an object



Solid angle $\Omega = A$ (area on unit sphere)

$= \frac{A'}{r^2}$ (area on any sphere)

Differential Solid angle $d\Omega$

$$d^3r = r^2 \sin\theta \, d\theta \, d\phi \, dr$$
$$= dA \, dr$$

dA = differential of area on sphere of radius r

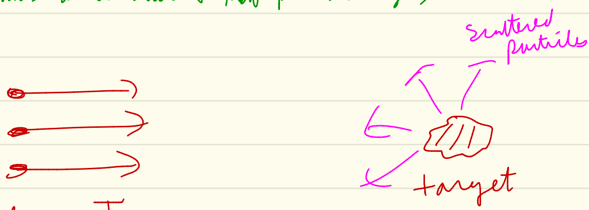
$$\boxed{\therefore d\Omega = \frac{dA}{r^2} = \sin\theta \, d\theta \, d\phi}$$

= range of directions cut out by $d\theta$ + $d\phi$ on unit sphere

$$\Omega_{max} = \int_0^\pi \sin\theta \, d\theta \int_0^{2\pi} d\phi = (2\pi)(-\cos\theta)\Big|_0^\pi = 4\pi \quad \text{(area of unit sphere)}$$

Scattering Cross Section (statistical description of collisions)

* Describes capability of a target to subject the incident particles to some fate (e.g., scatter into some direction. In QM, more exotic "fates" possible, e.g., incident particles can be absorbed + "new" particles emerge.)



Incident flux $\underset{\sim}{I}$

$$I = \frac{\#\text{ of incident particle}}{\text{time}\cdot\text{transverse area}}$$

* Let $dN \equiv$ # of projectile particles meeting some fate due to interaction w/target (e.g., getting deflected into some range of directions)

* Physically, expect $dN \propto I$

$$dN = I\,\underline{d\sigma}$$

# of particles/time meeting some fate (e.g., getting deflected in some range of directions)

coefficient of proportionality "cross section" ($[d\sigma]$ = area)

$$dN = I\,d\sigma = I\,\frac{d\sigma}{d\Omega}\,d\Omega$$

if the "fate" is to be scattered into some range of directions specified by $d\Omega$.

$$\Rightarrow \frac{dN}{d\Omega} = I\,\frac{d\sigma}{d\Omega}$$

\# deflected into $d\Omega$ about some direction per time

"Differential cross section"

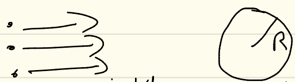
* <u>Total scattering cross section</u>

$$\sigma_t = \int d\sigma = \int \frac{d\sigma}{d\Omega}\,d\Omega = \frac{1}{I}\int dN = \frac{N}{I}$$

$\Rightarrow N = I\,\sigma_t$ = # incident particles per unit time affected (scattered) by the target

* If we are considering contact forces between target/projectile, then σ_t just common sense geometry

<u>ex1</u>:



point projectiles

spherical target

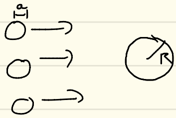
$$\Rightarrow \sigma_t = \pi R^2$$

Side remark: If we look at the quantum mechanical σ_t for the previous system, we find A) $\sigma_t^{qm} \neq \sigma_t^{cm}$, and moreover $\sigma_t^{qm} = \sigma_t^{qm}(E)$

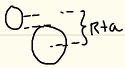
e.g., low E $\sigma_t^{qm} \to 4 \times \sigma_t^{cm}$ (de Broglie wavelength $\lambda = \frac{h}{p}$

QM effects make projectiles "effectively" bigger)

ex2: finite projectiles

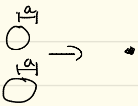


$\sigma_t \neq \pi R^2$



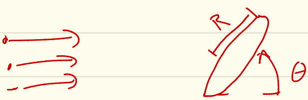
$\Rightarrow \sigma_t = \pi (R+a)^2$

ex3:



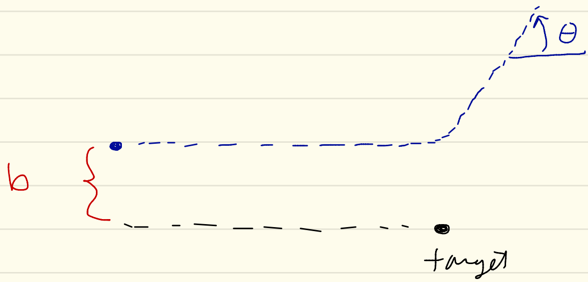
$\sigma_t = \pi a^2$

ex4:



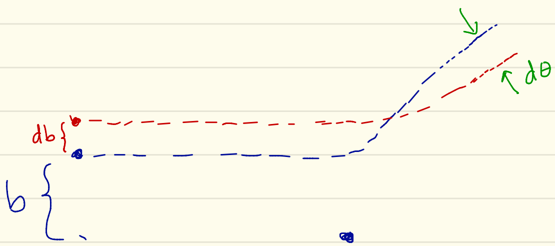
$\sigma_t = \pi R^2 \sin\theta$

* Scattering Cross Sections for Central Forces (no ϕ-dependence)



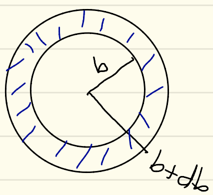
b = "impact parameter" (asymptotic $\perp$ distance between projectile & target)

* Clearly, we expect $\theta = \theta(b)$. Consider some db about b



NOTE: Here we make the reasonable assumption that the central force is such that deflection decreases w/ increasing b

i.e., $\frac{db}{d\theta} < 0$



View looking down the beam of projectiles

let $dN = \dfrac{\text{\# projectiles scattered thru } d\theta}{\text{sec}}$

$= I \cdot (2\pi b \, db)$ — area of shaded region

$= I \, d\sigma(\theta)$

so, $d\sigma = 2\pi b\, db = 2\pi b \left|\frac{db}{d\theta}\right| d\theta$

abs. value since by assumption $d\sigma > 0$ and $\frac{db}{d\theta} < 0$ and $d\theta > 0$.

$$\boxed{\Rightarrow \frac{d\sigma(\theta)}{d\theta} = 2\pi b \left|\frac{db}{d\theta}\right|}$$

* relating back to $\frac{d\sigma}{d\Omega}$ we use $d\Omega = d\phi \sin\theta\, d\theta = 2\pi \sin\theta\, d\theta$

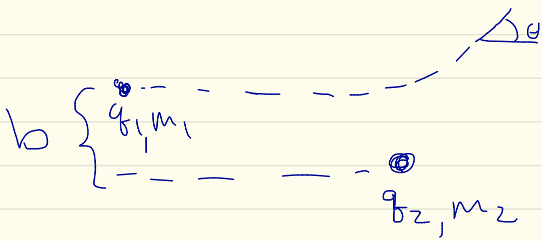
(axially symmetric)

$$\boxed{\therefore \frac{d\sigma(\theta)}{d\Omega} = \frac{1}{2\pi \sin\theta} \frac{d\sigma}{d\theta} = \frac{b}{\sin\theta} \left|\frac{db}{d\theta}\right|}$$

general result for differential cross section per solid angle for central force (i.e., $d\sigma(\theta,\phi) = d\sigma(\theta)$) & axially symmetric problems.

Example: Coulomb repulsion scattering

$$\vec{F} = k \frac{q_1 q_2}{r^3} \vec{r}$$



(assume $m_2 \gg m_1$)

Claim: $b(\theta) = \frac{Kq_1q_2}{2T} \cot\frac{\theta}{2}$ $T = \frac{1}{2}m_1v_1^2$

$$\frac{db}{d\theta} = \frac{Kq_1q_2}{4T} \frac{1}{\sin^2\frac{\theta}{2}}$$

$$\therefore \frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right| = \frac{Kq_1q_2}{2T}\frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} \times \frac{1}{\sin\theta}\frac{Kq_1q_2}{4T}\frac{1}{\sin^2\frac{\theta}{2}}$$

$$= \frac{Kq_1q_2}{2T}\frac{\cancel{\cos\frac{\theta}{2}}}{\sin\frac{\theta}{2}} \times \frac{1}{2\sin\frac{\theta}{2}\cancel{\cos\frac{\theta}{2}}} \times \frac{Kq_1q_2}{4T}\frac{1}{\sin^2\frac{\theta}{2}}$$

$$= \frac{1}{16}\left(\frac{Kq_1q_2}{T}\right)^2 \frac{1}{\sin^4\frac{\theta}{2}}$$ "Rutherford cross section"

Remarkably, in a lucky accident due to the $\frac{1}{r^2}$ form of the Coulomb force, the CM + QM expressions agree. This allowed Rutherford to deduce the presence of the atomic nucleus in α-Gold scattering.