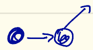
# New topic: Collisions

* collision = an interaction process that lasts a very short time and alters the course of interacting objects

e.g. billiard balls colliding $\Longleftrightarrow$ contact interaction 

* During collision, the forces act over a very short period of time $\Delta t$ ("Impulsive Forces")

* Even if we're ignorant of the precise form of $\vec{F}$ during the collision, Newton's 2nd Law still holds

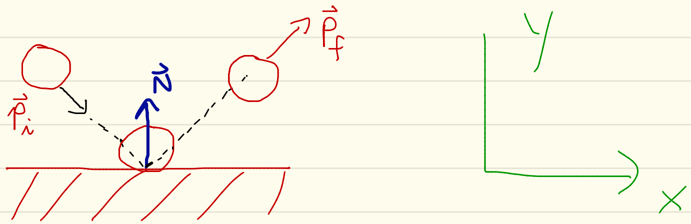
$$\vec{F} = \frac{d}{dt}(m\vec{v})$$

$$\underbrace{\int_0^{\Delta t} \vec{F}\, dt}_{\text{"Impulse"}} = \int d\vec{p} = \vec{p}(\Delta t) - \vec{p}(0)$$

* Since $\Delta t$ "small", to good approx. we have

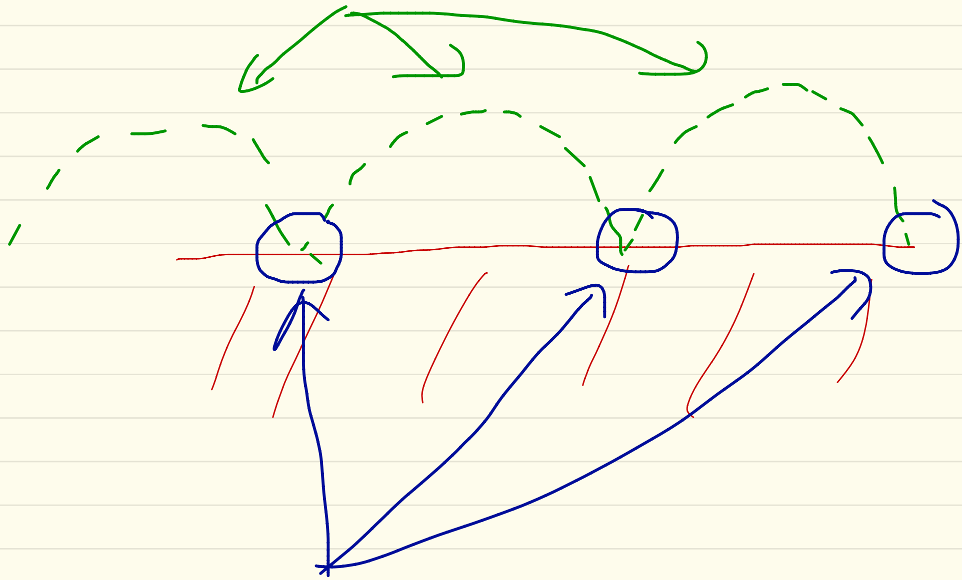
$$\Delta\vec{p} \simeq \vec{F}\Delta t \quad \Rightarrow \quad \vec{F} \simeq \frac{\overset{\text{finite}}{\Delta\vec{p}}}{\underset{\text{very short}}{\Delta t}}$$

$\Rightarrow$ collisional forces tend to be huge so that forces external to the interacting objects may be ignored during the collision eg. A bouncing ball



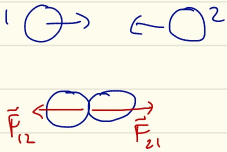

$$\vec{p}_f - \vec{p}_i = \int_0^{\Delta t} \vec{N}\, dt \simeq \vec{N} \Delta t \Rightarrow (\Delta \vec{p}) \cdot \hat{i} = 0$$ (horiz. $\vec{p}$-component unchanged)

$\vec{N} = \frac{\Delta \vec{p}}{\Delta t}$ = very large $\Rightarrow$




gravity unimportant here compared to $\vec{N}$

# Collision between 2 bodies




$$\vec F_{12} = -\vec F_{21} \Rightarrow \frac{d}{dt}\vec p_1 = -\frac{d}{dt}\vec p_2$$

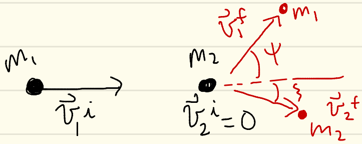
$$\Rightarrow \frac{d}{dt}(\vec p_1 + \vec p_2) = 0$$

$$\Rightarrow \vec p_1^{\,i} + \vec p_2^{\,i} = \vec p_1^{\,f} + \vec p_2^{\,f} \quad \Rightarrow \vec P = \text{const}$$

* Elastic collisions => $T = \text{const}$ too

* 2 Convenient Reference Frames to analyze the problem

## LAB

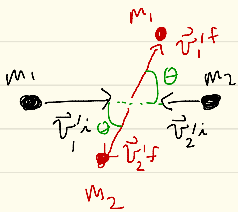



$$\vec P = \text{const} \Rightarrow \vec P^i = \vec P^f$$

x: $m_1 v_1^i = m_1 v_{1f} \cos\psi + m_2 v_2^f \cos\xi$

y: $0 = m_1 v_1^f \sin\psi - m_2 v_2^f \sin\xi$

## CM




$$\vec P' = \text{const} = 0 \Rightarrow$$

$$\vec p_1'^{\,i} = -\vec p_2'^{\,i}, \quad \vec p_1'^{\,f} = -\vec p_2'^{\,f}$$

## Elastic Collisions $\Rightarrow$ $T = \text{const} \Rightarrow T_f = T_i$

LAB: $T = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2}$

* use $\vec{v}_\alpha = \vec{V} + \vec{v}_\alpha'$

$$\Rightarrow \quad T = \underbrace{\frac{MV^2}{2}}_{\text{CM motion}} + \underbrace{T'}_{\text{motion about CM}}$$

$$T' = \frac{p_1'^2}{2m_1} + \frac{p_2'^2}{2m_2}$$ * but $\vec{p}_1' = -\vec{p}_2' \equiv \vec{p}'$

$$T' = \frac{p'^2}{2}\left(\frac{1}{m_1} + \frac{1}{m_2}\right)$$ $\Rightarrow$ define Reduced Mass $\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}$

$$\boxed{\therefore \; T' = \frac{p'^2}{2\mu}}$$

*Note: define relative velocity $\vec{v} = \vec{v}_1' - \vec{v}_2' = \vec{v}_1 - \vec{v}_2$ (Coord. System independent)

$$\boxed{\vec{v} = \frac{\vec{p}_1'}{m_1} - \frac{\vec{p}_2'}{m_2} = \frac{\vec{p}'}{m_1} + \frac{\vec{p}'}{m_2} = \frac{\vec{p}'}{\mu} \quad \text{(like a single particle of mass } \mu)}$$

$\Downarrow$

$$\boxed{T' = \frac{1}{2}\mu v^2}$$

relative motion behaves like a single particle of mass $\mu$.

$$T = \frac{1}{2}MV^2 + \frac{1}{2}\mu v^2$$

$$= \frac{P^2}{2M} + \frac{p'^2}{2\mu} \qquad \text{where } \vec{v} = \vec{v}_1 - \vec{v}_2 = \vec{v}_1{}' - \vec{v}_2{}'$$

and

$$\vec{p}\,' = \mu \vec{v}$$

$$\frac{1}{\mu} \equiv \frac{1}{m_1} + \frac{1}{m_2} = \frac{M}{m_1 m_2}$$

Now, $\vec{P} = \text{const}$ in a collision process (ALWAYS!)

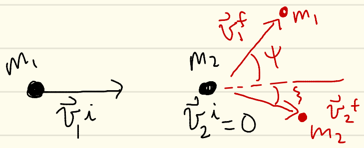
and $T = \text{const}$ in elastic collisions

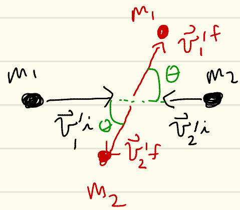
$$\Rightarrow T^i = T^f \Rightarrow \cancel{\frac{P^2}{2M}} + \frac{p_i'^2}{2\mu} = \cancel{\frac{P^2}{2M}} + \frac{p_f'^2}{2\mu}$$

$$\Rightarrow \frac{\mu v_i^2}{2} = \frac{\mu v_f^2}{2}$$

$$\boxed{\begin{aligned} \Rightarrow \quad v_i &= v_f \\ p_i' &= p_f' \end{aligned}}$$

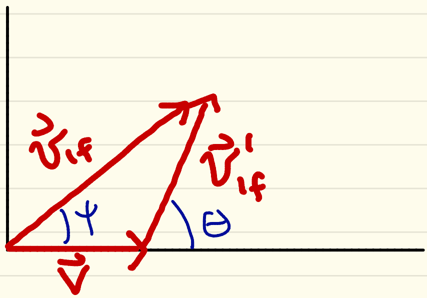
# Relating LAB | CM quantities




* Find relation between $\psi$ & $\theta$

use $\vec{v}_1{}^f = \vec{V} + \vec{v}\,'_{1f}$




$\underline{y}$: $v_{1f}^{y} = v_{1f}^{\prime y}$

$\Rightarrow v_{1f} \sin\psi = v'_{1f} \sin\theta$ ①

$\underline{x}$: $v_{1f}^{x} = V + v_{1f}^{\prime x}$

$\Rightarrow v_{1f} \cos\psi = V + v'_{1f} \cos\theta$ ②

$\frac{①}{②} \Rightarrow \tan\psi = \frac{v'_{1f}\sin\theta}{V + v'_{1f}\cos\theta}$

$$\boxed{\tan\psi = \frac{\sin\theta}{\frac{V}{v'_{1f}} + \cos\theta}}$$

$$\boxed{\tan\psi = \frac{\sin\theta}{\frac{V}{v'_{1f}} + \cos\theta}}$$

* but $\vec{P} = M\vec{V} = m_1 \vec{v}_{1i}$

$$\Rightarrow \boxed{V = \frac{m_1}{m_2+m_1} v_{1i}} \quad (a)$$

$v_i^{rel} = v_f^{rel}$

$$|\vec{v}'_{1i} - \vec{v}'_{2i}| = |\vec{v}_{1i} - \cancel{\vec{v}_{2i}}^{\,0}| = |\vec{v}'_{1f} - \vec{v}'_{2f}|$$

* but $\vec{p}'_{1f} = -\vec{p}'_{2f} \Rightarrow \vec{v}'_{2f} = -\frac{m_1}{m_2}\vec{v}'_{1f}$

$$\boxed{\therefore v_{1i} = \left|\left(1+\frac{m_1}{m_2}\right)\vec{v}'_{1f}\right| = \left(1+\frac{m_1}{m_2}\right)v'_{1f}} \quad (b)$$

(a) & (b) combine to give

$$V = \left(1+\frac{m_1}{m_2}\right)\left(\frac{m_1}{m_2+m_1}\right) v'_{1f}$$

$$\boxed{\therefore \frac{V}{v'_{1f}} = \frac{\cancel{(m_2+m_1)}\, m_1}{\cancel{(m_2+m_1)}\, m_2} = \frac{m_1}{m_2}}$$

* This gives the desired relationship

$$\boxed{\tan\psi = \frac{\sin\theta}{\frac{m_1}{m_2} + \cos\theta}}$$