Name: Solution Key **Student ID:** ______________

S13 PHY321: Final

May 1, 2013

NOTE: Show all your work. *No credit* for unsupported answers.

Turn the front page only when advised by the instructor!

The exam consists of 6 problems (60 points) plus one extra credit problem (10 points).

Contextual information

Damped harmonic oscillator equation: $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$

General solution: $x(t) = \mathrm{e}^{-\beta t}\left[A_1 \exp\left(\sqrt{\beta^2 - \omega_0^2}t\right) + A_2 \exp\left(-\sqrt{\beta^2 - \omega_0^2}t\right)\right]$

Driven harmonic oscillator equation: $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = A\cos\omega t$

Amplitude of stationary oscillations: $D = \frac{A}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + 4\omega^2\beta^2}}$

Phase lag: $\delta = \tan^{-1}\left(\frac{2\omega\beta}{\omega_0^2 - \omega^2}\right)$

Acceleration vector in polar coordinate: $\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\mathbf{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\mathbf{e}_\theta$

$G = 6.673 \times 10^{-11} \frac{\mathrm{m}^3}{\mathrm{kg\,s}^2}$

Calculus in spherical coordinates:

$\vec{\mathrm{grad}}\,\Psi = \vec{\mathrm{e}}_r \frac{\partial\Psi}{\partial r} + \vec{\mathrm{e}}_\theta \frac{1}{r}\frac{\partial\Psi}{\partial\theta} + \vec{\mathrm{e}}_\phi \frac{1}{r\sin\theta}\frac{\partial\Psi}{\partial\phi}$

$\vec{\mathrm{div}}\,\vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\, A_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$

$\nabla^2\Psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Psi}{\partial\phi^2}$

Orbits in a gravitational field:

$$\frac{\alpha}{r} = 1 + \epsilon\cos\theta \qquad a = \frac{k}{2|E|} \qquad b = \frac{\ell}{\sqrt{2\mu|E|}} \qquad \alpha = \frac{\ell^2}{\mu k}$$

1. Consider the two forces

$$\vec{F}_A = (2y, x, 0)\,, \qquad \textit{and} \qquad \vec{F}_B = (y+1, x, 0)\,.$$

where the force and position components are, respectively, in newtons and meters.

(a) [2 pts] Find magnitude of the force $\vec{F}_B$ at the location $(x, y, z) = (-2, 3, 0)$.

$\vec{F}_B(-2,3,0) = 4\hat{\imath} - 2\hat{\jmath} \Rightarrow |\vec{F}_B| = \sqrt{16+4} = \sqrt{20} = 4.47$ Newtons

(b) [3 pts] Find the angle that the force $\vec{F}_B$ makes with the positive direction of the x-axis at the location $(x, y, z) = (-2, 3, 0)$.

$F_{B,x} = \vec{F}_B \cdot \hat{\imath} = |\vec{F}_B| \cos\theta \Rightarrow \cos\theta = \dfrac{F_{B,x}}{|\vec{F}_B|} = \dfrac{4}{4.47} = .895$

$\Rightarrow \theta = \cos^{-1}\left(\frac{4}{4.47}\right) = 26.49^\circ$

(c) [5 pts] Are either of the forces conservative? If so, determine the corresponding potential(s).

If $\vec{\nabla}\times\vec{F} = 0$, then $\vec{F}$ is conservative & can be written $\vec{F} = -\vec{\nabla} U$

$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ F_x & F_y & F_z \end{vmatrix} = \hat{\imath}(\partial_y F_z - \partial_z F_y) + \hat{\jmath}(\partial_z F_x - \partial_x F_z) + \hat{k}(\partial_x F_y - \partial_y F_x)$

* we see $\vec{\nabla}\times\vec{F}_A \neq 0$ while $\vec{\nabla}\times\vec{F}_B = \hat{\imath}(0-0) + \hat{\jmath}(0-0) + \hat{k}(1-1) = 0$

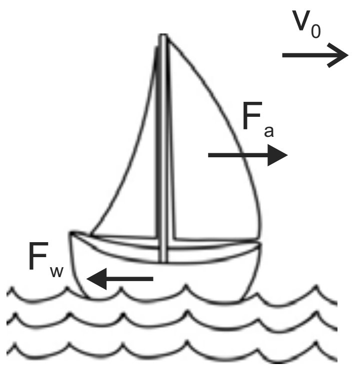
$\therefore$ $\vec{F}_B$ is conservative. We find it using:

① $F_x = -\dfrac{\partial U}{\partial x} = y+1 \Rightarrow U(x,y) = -xy - x + C(y)$

② $F_y = -\dfrac{\partial U}{\partial y} = x = x - \dfrac{\partial C}{\partial y} \Rightarrow C = \text{constant}$

$$\boxed{\therefore U = -x(y+1) + \text{const}}$$

2. A sailboat of mass m is moving on water, in the direction of wind. The sailboat is subject to two forces in the horizontal direction: $\vec{F} = \vec{F}_a + \vec{F}_w$, where $\vec{F}_a$ is due to air and $\vec{F}_w$ is due to water. The force of air is propelling the sailboat and it is proportional to the difference between the constant velocity of the wind $\vec{v}_0$ and the boat velocity $\vec{v}$, $\vec{F}_a = A(\vec{v}_0 - \vec{v})$. The force of water resists the motion of the boat and is proportional to the boat velocity, $\vec{F}_w = -B\,\vec{v}$. Here, A and B are positive proportionality constants.



(a) [2 pt] For what velocity $\vec{v}$ of the boat is the net horizontal force on the boat going to be zero?

(b) [8 pts] Find the dependence of the boat's velocity v on time t, when the boat starts from rest at $t = 0$.

By taking the limit $t \to \infty$, verify that the expression you found is consistent with the answer to (a).

a) $\vec{F} = 0 = A(\vec{v}_0 - \vec{v}) - B\vec{v} \Rightarrow A\vec{v}_0 - \vec{v}(A+B) = 0$

$$\boxed{\therefore\ \vec{v} = \frac{A}{A+B}\vec{v}_0}$$

b) $m\frac{dv}{dt} = -Bv + A(v_0 - v) = -(A+B)v + Av_0$

$\Rightarrow \frac{dv}{dt} + \left(\frac{A+B}{m}\right)v = \frac{A}{m}v_0$ IHG eqn $\Rightarrow v = v_{HG} + v_{IHG}$

where: $\frac{d}{dt}v_{HG} = -\left(\frac{A+B}{m}\right)v_{HG} \Rightarrow v_{HG} = Ce^{-\frac{A+B}{m}t}$ C TBD from $v(0) = 0$

* per the trick discussed in class, we try to "guess" v_{IHG} by assuming $\frac{d}{dt}v_{IHG} = 0$

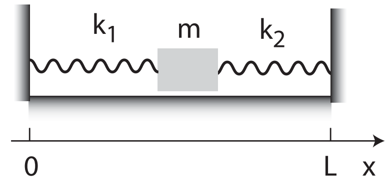
$\Rightarrow \left(\frac{A+B}{m}\right)v_{IHG} = \frac{A}{m}v_0 \Rightarrow v_{IHG} = \frac{A}{A+B}v_0$

$\therefore v = Ce^{-\frac{A+B}{m}t} + \frac{A}{A+B}v_0$ * Demanding $v(0) = 0 \Rightarrow C = -\frac{A}{A+B}v_0$

$$\boxed{\Rightarrow v = \frac{A}{A+B}v_0\left(1 - e^{-\frac{A+B}{m}t}\right)}$$

$$\boxed{\text{check: } v(\infty) = \frac{A}{A+B}v_0 \text{ as in part a) } \checkmark}$$

3. A small object of mass m is positioned on a smooth plane and attached to walls on its two sides, at $x = 0$ and $x = L$, with stretched massless horizontal springs of spring constants k_1 and k_2, respectively, as displayed in the figure. The unstretched lengths of both springs are negligible.



(a) [2 pts] Find the equilibrium position x_0 for the mass.

$F_1 = -k_1 x_0$

$F_2 = k_2(L - x_0)$

$\Big\} \; F = F_1 + F_2 = 0 = -x_0(k_1+k_2) + k_2 L$

$$\Rightarrow \boxed{x_0 = \frac{k_2}{k_1+k_2} L}$$

(b) [2 pts] What is the net force acting on the mass in the x direction, if the mass is at a location x, $x \neq x_0$ and $0 < x < L$?

$F_1 = -k_1 x$

$F_2 = k_2(L - x)$

$F = F_1 + F_2 = -(k_1+k_2)x + k_2 L$

$*$ but $L = x_0 \frac{k_1+k_2}{k_2}$

$= -(k_1+k_2)x + (k_1+k_2)x_0$

$$\boxed{F = -(k_1+k_2)(x - x_0)}$$

(c) [4 pts] What is the angular frequency of oscillations about the equilibrium position, in absence of friction? Compute the value for $m = 2.50\,\text{kg}$, $k_1 = 3.50\,\text{N/m}$, $k_2 = 1.50\,\text{N/m}$ and $L = 0.90\,\text{m}$. Are all constants needed?

$$\boxed{\begin{array}{l} \omega_0 = \sqrt{\frac{k_{eff}}{m}} \quad \text{where } k_{eff} = k_1 + k_2 \text{ (see part b)} \\ = \sqrt{\frac{3.5+1.5}{2.5}} = \sqrt{2} = 1.41 \ \frac{\text{rad}}{\text{s}} \end{array}}$$

(d) [2 pt] If a friction force acts on the mass, opposite and proportional to velocity, $F_f = -b\,\dot{x}$, with a proportionality constant of $b = 0.80\,\text{kg/s}$, is the motion underdamped, overdamped or approximately critically damped?

$$\beta = \frac{b}{2m} = \frac{.8}{5} = .16 < \omega_0$$

$\Rightarrow$ <u>Underdamped</u>.

4. A spherical planet of radius R has a density ρ that is the largest at its center, and decreases with distance r from the center as $\rho(r) = A\,(2R - r)$, where A is a constant with the appropriate units.

(a) [4 pts] Determine the mass of the planet.

$$M = \int \rho(r)\, d^3r = 4\pi \int_0^R \rho(r)\, r^2 dr$$

$$= 4\pi A \int_0^R (2Rr^2 - r^3)\, dr = 4\pi A \left[\frac{2}{3} R^4 - \frac{R^4}{4} \right]$$

$$= \frac{4\pi A}{12} \cdot \left[8R^4 - 3R^4 \right] = \boxed{\frac{5\pi A}{3} R^4 = M}$$

(b) [6 pts] Determine the gravitational field at distance r from the center of the planet.

$$\oint_S \vec{g} \cdot d\vec{r} = -4\pi G M_{enc} \quad \text{* take } S = \text{sphere radius } r$$

$$g(r)\cancel{4\pi} r^2 = -\cancel{4\pi} G M_{enc}(r)$$

$$\boxed{\Rightarrow g(r) = -\frac{G M_{enc}(r)}{r^2}}$$

* for $r > R$, $M_{enc}(r) = M$ from part a).

* for $r < R$, $M_{enc}(r) = 4\pi A \left[\frac{2}{3} R r^3 - \frac{r^4}{4} \right]$

5. A satellite is in a circular orbit of radius R around Earth (mass $M = 5.97 \times 10^{24}$ kg).

(a) [2 pt] How is the velocity v of the satellite related to the radius R, mass M and gravitational constant G?

$$\frac{mv^2}{R} = \frac{GmM}{R^2}$$

$$\Rightarrow v^2 = \frac{GM}{R} \Rightarrow v = \sqrt{\frac{GM}{R}}$$

(b) [4 pts] What needs to be the radius R to make the orbit semisynchronous, i.e. with a period of 12h? (GPS satellites move on such orbits.) Obtain a value for R.

$$\tau = \frac{2\pi R}{v} = 2\pi R \times \sqrt{\frac{R}{GM}} = \frac{2\pi R^{3/2}}{\sqrt{GM}}$$

$$R^{3/2} = \frac{\sqrt{GM}\,\tau}{2\pi} \Rightarrow R = \left(\frac{\sqrt{GM}\,\tau}{2\pi}\right)^{2/3} = 26{,}601 \text{ km}$$

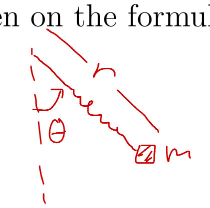
(c) [4 pts] Now imagine that the satellite has a parabolic orbit. If the satellite has the same angular momentum as it did for the circular orbit, how big is r_{min} (i.e., the perihelion) compared to R in part a.

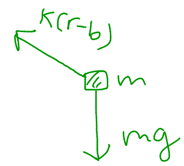
$$\frac{\alpha}{r_{min}} = 1 + \varepsilon$$

$$\therefore r_{min} = \frac{\alpha}{2} = \frac{R}{2} \quad (\text{recall, } R = \alpha)$$

6. Consider a pendulum consisting of a mass m suspended by a massless spring. The spring has a unextended length of b, and a spring constant k. The pendulum is in a uniform gravitational field g.

(a) [5pts] Using Newton's formulation of mechanics (i.e., $\mathbf{F} = m\mathbf{a}$), find equations of motion for the mass in terms of the variables r and θ. (Note: The acceleration vector in polar coordinates is given on the formula sheet.)





$$F_r = mg\cos\theta - k(r-b)$$
$$F_\theta = -mg\sin\theta$$

$$\therefore F_r = ma_r \Rightarrow m(\ddot{r} - r\dot{\theta}^2) = mg\cos\theta - k(r-b)$$
$$\Rightarrow \boxed{\ddot{r} - r\dot{\theta}^2 - g\cos\theta + \frac{k}{m}(r-b) = 0}$$

$$F_\theta = ma_\theta$$
$$= \cancel{m}(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = -\cancel{m}g\sin\theta \Rightarrow \boxed{\ddot{\theta} + 2\frac{\dot{r}}{r}\dot{\theta} + \frac{g}{r}\sin\theta = 0}$$

(b) [5pts] Using the Lagrangian formulation of mechanics, find the Euler-Lagrange equations for r and θ. Should they agree with part a)?

$$T = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) \quad \text{and} \quad U = -mgr\cos\theta + \frac{1}{2}k(r-b)^2$$
$$\Rightarrow L = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) + mgr\cos\theta - \frac{1}{2}k(r-b)^2$$

EL in r: $\left.\begin{aligned} \frac{\partial L}{\partial r} &= mr\dot{\theta}^2 + mg\cos\theta - k(r-b) \\ \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} &= m\ddot{r} \end{aligned}\right\} \Rightarrow \boxed{\ddot{r} - r\dot{\theta}^2 - g\cos\theta + \frac{k}{m}(r-b) = 0}$

EL in θ: $\left.\begin{aligned} \frac{\partial L}{\partial \theta} &= -mgr\sin\theta \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) &= \frac{d}{dt}(mr^2\dot{\theta}) = mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} \end{aligned}\right\} \Rightarrow mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} + mgr\sin\theta = 0$

$$\Rightarrow \boxed{\ddot{\theta} + 2\frac{\dot{r}\dot{\theta}}{r} + \frac{g}{r}\sin\theta = 0}$$

*It agrees w/part a) as it should.

7. **[Extra Credit]** A simple pendulum consists of a massless rod of fixed length b with a mass m attached to the end. There is a uniform gravitational field g pointing downwards. Starting at $t = 0$, the pendulum is moved upward with uniform acceleration a.

(a) [4 pts] Write down the Lagrangian in the appropriate set of generalized coordinates. (Hint: Start in cartesian coordinates, and write x, y in terms of θ, b, a, and t.)

$$\left.\begin{aligned} x &= b\sin\theta \\ y &= \frac{at^2}{2} - b\cos\theta \end{aligned}\right\} \Rightarrow \begin{aligned} \dot{x} &= b\dot{\theta}\cos\theta \\ \dot{y} &= at + b\dot{\theta}\sin\theta \end{aligned}$$

$$T = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) = \frac{m}{2}\left(b^2\dot{\theta}^2\cos^2\theta + a^2t^2 + b^2\dot{\theta}^2\sin^2\theta + 2atb\dot{\theta}\sin\theta\right)$$

$$\Rightarrow T = \frac{m}{2}\left(b^2\dot{\theta}^2 + a^2t^2 + 2atb\dot{\theta}\sin\theta\right)$$

$$U = mgy = mg\left(\frac{at^2}{2} - b\cos\theta\right)$$

$$\boxed{\Rightarrow L = T - U = \frac{m}{2}\left(b^2\dot{\theta}^2 + a^2t^2 + 2atb\dot{\theta}\sin\theta\right) - mg\left(\frac{at^2}{2} - b\cos\theta\right)}$$

(b) [3 pts] Derive the Euler-Lagrange equation of motion.

$$\frac{\partial L}{\partial \theta} = matb\dot{\theta}\cos\theta - mgb\sin\theta$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{d}{dt}\left[mb^2\dot{\theta} + matb\sin\theta\right] = mb^2\ddot{\theta} + matb\dot{\theta}\cos\theta + mab\sin\theta$$

$$\Rightarrow mb^2\ddot{\theta} + \cancel{matb\dot{\theta}}\cos\theta + mab\sin\theta = \cancel{matb\dot{\theta}}\cos\theta - mgb\sin\theta$$

$$\Rightarrow \boxed{\ddot{\theta} + \left(\frac{a+g}{b}\right)\sin\theta = 0}$$

(c) [3 pts] Find the period of oscillations in the limit of small θ.

$$\sin\theta \approx \theta \Rightarrow \ddot{\theta} + \left(\frac{a+g}{b}\right)\theta = 0$$

$$\Rightarrow \omega_0^2 = \frac{a+g}{b}$$

$$\Rightarrow \boxed{T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{b}{a+g}}}$$

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